An Extended Framework for Specifying and Reasoning about Proof Systems

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Abstract

It has been shown that linear logic can be successfully used as a framework for both specifying proof systems for a number of logics, as well as proving fundamental properties about the specified systems. In this paper, we show how to extend the framework with subexponentials in order to be able to declaratively encode a wider range of proof systems, including a number of non-trivial proof systems such as a multi-conclusion intuitionistic logic, classical modal logic S4, and intuitionistic Lax logic. Moreover, we propose methods for checking whether an encoded proof system has important properties, such as if it admits cut-elimination, the completeness of atomic identity rules, and the invertibility of its inference rules. Finally, we present a tool implementing some of these specification/verification methods.

1 Introduction

Designing suitable proof systems for specific applications has become one of the main tasks of many applied logicians working in computer science. Proof theory has been applied in different fields including programming languages, knowledge representation, automated reasoning, access control, among many others. It is of utmost importance to guarantee that such designed proof systems have *good properties*, *e.g.* the admissibility of the cut-rule (which leads to other important properties such as the sub-formula property and the consistency of the system) as well as the completeness of atomic identity rules and the invertibility of inference rules. It is therefore of interest to develop techniques and *automated* tools that can help logicians (and possibly non-logicians) in specifying and reasoning about proof systems.

In the recent years, a series of papers [18, 17, 22, 27] have shown that linear logic [11] can be used as a framework for *specifying* and *reasoning* about proof systems. In particular, [27, 22] showed how to specify not only sequent calculus systems, but also natural deduction systems for different logics, such as minimal, intuitionistic and classical logics. Moreover, in [18, 17] it is shown how to check whether an encoded proof system enjoys important properties by simply analyzing its linear logic specification. For instance, in those works, sufficient conditions are provided for guaranteeing cut-elimination for specified systems.

In our previous work [23], we proposed using linear logic with *subexponentials* as a framework for specifying proof systems. The motivation for this step comes from the fact that, since exponentials in linear logic are not canonical [20, 7], one can construct linear logic proof systems containing as many subexponentials as one needs. Such subexponentials may or may not allow contraction and weakening. Subexponentials therefore allow for the specification of systems with multiple contexts, which may be represented by sets or multisets of formulas. These features made it possible to declaratively encode a wide range of proof systems, such as multi-conclusion proof system for intuitionistic logic. And, since the proposed encoding is natural and direct, we were able to use the rich linear logic meta-level theory in order to reason about the specified systems in an elegant and simple way.

The contribution of this paper is three-fold. First, in Section 4, we demonstrate how to declaratively specify proof systems with more involved structural and logical inferences rules using linear logic theories with subexponentials. We encode proof systems that have structural restrictions that are much more interesting and challenging than of the systems specified in [23]. Besides the multi-conclusion system for intuitionistic logic specified in our previous work, we specify proof systems for intuitionistic lax logic [9], focused intuitionistic logic *LJQ** and classical modal logic S4. These examples provide evidence that linear logic with subexponentials can be successfully used as a framework for a number of proof systems for modal and focused logics.

Our second contribution, in Section 5, follows and enhances the ideas presented in [18]. We provide sufficient conditions for guaranteeing three properties for systems specified using subexponentials: (1) the admissibility of the cut-rule; (2) the completeness of the system when only using atomic instances of the initial rule; and (3) for determining whether an inference rule is invertible. The main difference from what is presented here and the work developed in [18] is the establishment of some criteria for *permutation of rules*. Such analysis is needed for checking whether proofs with cuts can be transformed into proofs with *principal cuts*. Since our framework enables for the encoding of much more complicated proof systems, the behavioral analysis is more involved and it leads to more general conditions when compared to [18].

Finally, we have implemented a tool, described in Section 6, that accepts a linear logic specification with subexponentials and automatically checks whether principals cuts can be reduced to atomic cuts and whether initial rules can be atomic only. Our tool is able to show that all the systems mentioned above satisfy these conditions. Furthermore it also can check cases for when the cut-rule can be permuted over an introduction rule and when an introduction rule can permute over another introduction rule. Such analysis can greatly help to discover corner cases for when the reduction of a proof with cuts into a proof with principal cuts only is not immediate.

This paper is structured as follows. Section 2 introduces the proof system for linear logic with subexponentials, called *SELLF*, which is the basis of the proposed logical framework. In Section 3, we describe how to encode a proof system in our framework. Section 4 describes the encoding of a number of proof systems, namely, the proof system $G1m$ for minimal logic [29], multi-conclusion proof system for intuitionistic logic *mLJ* [15], the focused proof system *LJQ** for intuitionistic logic [8], a proof system for classical modal logic S4, and a proof system for intuitionistic lax logic [9]. Section 5 introduces the conditions for verifying whether an encoded proof system satisfies the properties mentioned before, which can be checked using our tool described in Section 6. Finally, in Sections 7 and 8, we end by discussing related and future work.

This is an improved and expanded version of the workshop paper [23].

2 Linear Logic with Subexponentials

Although we assume that the reader is familiar with linear logic, we review some of its basic proof theory. *Literals* are either atomic formulas (A) or their negations ($A^{\perp}$). The connectives $\otimes$ and $⅋$ and their units 1 and $\perp$ are *multiplicative*; the connectives $\oplus$ and $\&$ and their units 0 and $\top$ are *additive*; $\forall$ and $\exists$ are (first-order) quantifiers; and ! and ? are the exponentials. We shall assume that all formulas are in *negation normal form*, meaning that all negations have atomic scope.

Due to the exponentials, one can distinguish in linear logic two kinds of formulas: the linear ones whose main connective is not a ? and the unbounded ones whose main connective is a ?. The linear formulas can be seen as resources that can only be used once, while the unbounded formulas represent unlimited resources that can be used as many times as necessary. This distinction is usually reflected in syntax by using two different contexts in linear logic sequents ($\vdash \Theta : \Gamma$), one (Θ) containing only unbounded formulas and another (Γ) with only linear formulas [1]. Such distinction allows to incorporate structural rules, *i.e.*, weakening and contraction, into the introduction rules of connectives, as done in similar presentations for classical logic, *e.g.*, the *G3c* system in [29]. In such presentation, the context (Θ) containing unbounded formulas is treated as a set of formulas, while the other context (Γ) containing only linear formulas is treated as a multiset of formulas.

It turns out that the exponentials are not canonical [7] with respect to the logical equivalence relation. In fact, if, for any reason, we decide to define a blue and red conjunctions ($\wedge^b$ and $\wedge^r$ respectively) with the standard classical rules:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge^b B \vdash \Delta}\ [\wedge^b L] \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge^b B}\ [\wedge^b R]$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge^r B \vdash \Delta}\ [\wedge^r L] \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge^r B}\ [\wedge^r R]$$

then it is easy to show that, for any formulas A and B, $A \wedge^b B \equiv A \wedge^r B$. This means that all the symbols for classical conjunction belong to the same equivalence class. Hence, we can choose to use as the conjunction's *canonical* form any particular color, and provability is not affected by this choice. However, the same behavior does not hold with the linear logic exponentials. In fact, suppose we have red $!^r, ?^r$ and blue $!^b, ?^b$ sets of exponentials with the standard linear logic rules:

$$\frac{\vdash ?^r\Gamma, F}{\vdash ?^r\Gamma, !^r F}\ [!^r] \qquad \frac{\vdash \Gamma, F}{\vdash \Gamma, ?^r F}\ [D?^r] \qquad \frac{\vdash ?^b\Gamma, F}{\vdash ?^b\Gamma, !^b F}\ [!^b] \qquad \frac{\vdash \Gamma, F}{\vdash \Gamma, ?^b F}\ [D?^b]$$

We cannot show that $!^r F \equiv !^b F$ nor $?^r F \equiv ?^b F$. This opens the possibility of defining classes of exponentials, called *subexponentials* [21]. In this way, it is possible to build proof systems containing as many exponential-like operators, ($!^l$, $?^l$) as one needs: they may or may not allow contraction and weakening, and are organized in a pre-order ($\preceq$) specifying the entailment relation between these operators. Formally, a proof system for linear logic with subexponentials, called $SELL_\Sigma$, is specified by a subexponential signature, Σ, of the form $\langle I, \preceq, \mathcal{U} \rangle$, where I is the set of labels for subexponentials, $\preceq$ is a preorder relation[1] among the elements of I, and $\mathcal{U} \subseteq I$, specifying which subexponentials allow for weakening

[1] A preorder relation is a binary relation that is reflexive and transitive.

and contraction. The preorder $\preceq$ is also assumed to be upwardly closed with respect to the set $\mathcal{U}$, that is, if $x \prec y$ and $x \in \mathcal{U}$, then $y \in \mathcal{U}$.[2]

For a given a subexponential signature Σ, the proof system $SELL_\Sigma$ contains the same introduction rules as in linear logic for all connectives, except the exponentials. These are specified, on the other hand, by the subexponential signature, Σ, as follows:[3]

$$\frac{\vdash C, \Delta}{\vdash ?^x C, \Delta}\ [D, \text{if } x \in I] \qquad \frac{\vdash ?^y C, ?^y C, \Delta}{\vdash ?^y C, \Delta}\ [C, \text{if } y \in \mathcal{U}] \qquad \frac{\vdash \Delta}{\vdash ?^z C, \Delta}\ [W, \text{if } z \in \mathcal{U}]$$

The first rule, called dereliction, can be applied to any subexponential, and contraction and weakening only to subexponentials that appear in the set $\mathcal{U}$. The promotion rule is given by the following inference rule:

$$\frac{\vdash ?^{x_1} C_1, \ldots, ?^{x_n} C_n, C}{\vdash ?^{x_1} C_1, \ldots, ?^{x_n} C_n, !^a C}\ [!^a]$$

where $a \preceq x_i$ for all $i = 1, \ldots, n$. The promotion rule will play an important role here, namely, to specify the structural restrictions of encoded proof systems. In particular, one can use a subexponential bang, $!^c$, to check whether there are only some type of formulas in the context, namely, those that are marked with subexponentials, $?^x$, such that $c \preceq x$. If there is any formula $?^y F$ in the context such that $c \npreceq y$, then $!^c$ cannot be introduced.

We classify all the subexponential indexes belonging to $\mathcal{U}$ as *unrestricted* or *unbounded*, and the remaining indexes as *restricted* or *bounded*.

Danos *et al.* showed in [7] that *SELL* admits cut-elimination.

Theorem 2.1

For any signature Σ, the cut-rule is admissible in $SELL_\Sigma$.

2.1 Focusing

First proposed by Andreoli [1] for linear logic, focused proof systems provide the normal form proofs for cut-free proofs. In this section, we review the focused proof system for *SELL*, called *SELLF*, proposed in [21].

In order to explain *SELLF*, we first recall some more terminology. We classify as *positive* the formulas whose main connective is either $\otimes, \oplus, \exists$, the subexponential bang, the unit 1 and positive literals. All other formulas are classified as *negative*. Figure 1 contains the focused proof system *SELLF* that is a rather straightforward generalization of Andreoli's original system. There are two kinds of arrows in this proof system. Sequents with the $\Downarrow$ belong to the *positive* phase and introduce the logical connective of the "focused" formula (the one to the right of the arrow): building proofs of such sequents may require non-invertible proof steps to be taken. Sequents with the $\Uparrow$ belong to the *negative* phase and decompose the formulas on their right in such a way that only invertible inference rules are applied. The structural rules $D_1, D_l, R\Uparrow$, and $R\Downarrow$ make the transition between a negative and a positive phase.

Similarly as in the usual presentation of linear logic, there is a pair of contexts to the left of $\Uparrow$ and $\Downarrow$ of sequents, written here as $\mathcal{K} : \Gamma$. The second context, Γ, collects the formulas whose main connective is not a question-mark, behaving as the bounded context in linear logic. But differently from linear logic, where the first context is a multiset of formulas whose main

[2] This last condition on the pre-order is necessary to prove that $SELL_\Sigma$ admits cut-elimination see [7].

[3] Whenever it is clear from the context, we will elide the subexponential signature Σ.

connective is a question-mark, we generalize $\mathcal{K}$ to be an *indexed context*, which is a mapping from each index in the set I (for some given and fixed subexponential signature) to a finite multiset of formulas, in order to accommodate for more than one subexponential in *SELLF*. In Andreoli's focused system for linear logic, the index set contains a single subexponential, ∞, and $\mathcal{K}[\infty]$ contains the set of unbounded formulas. Figure 2 contains different operations used in such indexed contexts. For example, the operation $(\mathcal{K}_1 \otimes \mathcal{K}_2)$, used in the tensor rule, specifies the resulting indexed context obtained by merging two contexts $\mathcal{K}_1$ and $\mathcal{K}_2$.

Focusing allows the composition of a collection of inference rules of the same polarity into a "macro-rule." Consider, for example, the formula $N_1 \oplus N_2 \oplus N_3$, where all N_1, N_2, and N_3 are negative formulas. Once focused on, the only way to introduce such a formula is by using a "macro-rule" of the form:

$$\frac{\vdash \mathcal{K} : \Gamma \Uparrow N_i}{\vdash \mathcal{K} : \Gamma \Downarrow N_1 \oplus N_2 \oplus N_3}$$

where $i \in \{1, 2, 3\}$. In this paper, we will encode proof systems in *SELLF* in such a way that the "macro-rules" available using our specifications match exactly the inference rules of the encoded system.

This paper will make great use of the promotion rule, $!^l$, in order to specify the structural restrictions of a proof system. In particular, this rule determines two different operations when seen from the conclusion to premise. The first one arises by its side condition: a bang can be introduced only if the linear contexts that are not greater to l are all empty. This operation is similar to the promotion rule in plain linear logic: a bang can be introduced only if the linear context is empty. The second operation is specified by using the operation $\mathcal{K} \leq_l$: in the premise of the promotion rule all unbounded contexts that are not greater than l are erased. Notice that such operation is not available in plain linear logic.

Nigam in [20] proved that *SELLF* is sound and complete with respect to *SELL*.

THEOREM 2.2
For any subexponential signature Σ, $SELLF_\Sigma$ is sound and complete with respect to $SELL_\Sigma$.

Finally, to improve readability, we will often show explicitly the formulas appearing in the image of the indexed context, $\mathcal{K}$, of a sequent. For example, if the set of subexponential indexes is $\{x1, \ldots, xn\}$, then the following negative sequent

$$\vdash \Theta_1 \vdots_{x1} \Theta_2 \vdots_{x2} \cdots \Theta_n \vdots_{xn} \Gamma \Uparrow L$$

denotes the *SELLF* sequent $\vdash \mathcal{K} : \Gamma \Uparrow L$, such that $\mathcal{K}[xi] = \Theta_i$ for all $1 \leq i \leq n$. We will also assume the existence of a maximal unbounded subexponential called ∞, which is greater than all other subexponentials. This subexponential is used to mark the linear logic specification of proof systems explained in the next section.

3 Encoding Proof Systems in *SELLF*

3.1 Encoding Sequents

Similar as in Church's simple type theory [5], we assume that linear logic propositions have type o and that the object-logic quantifiers have type $(term \to form) \to form$, where $term$ and $form$ are respectively the types for an object-logic term and for object-logic formulas. Moreover, following [26, 27, 22], we encode a sequent in *SELLF* by using two meta-level atoms $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ of type $form \to o$. These meta-level atoms are used to mark, respec-

$$\frac{\vdash \mathcal{K} : \Gamma \Uparrow L, A \quad \vdash \mathcal{K} : \Gamma \Uparrow L, B}{\vdash \mathcal{K} : \Gamma \Uparrow L, A \& B}\ [\&] \quad \frac{\vdash \mathcal{K} : \Gamma \Uparrow L, A, B}{\vdash \mathcal{K} : \Gamma \Uparrow L, A \parr B}\ [\parr] \quad \frac{}{\vdash \mathcal{K} : \Gamma \Uparrow L, \top}\ [\top]$$

$$\frac{\vdash \mathcal{K} : \Gamma \Uparrow L}{\vdash \mathcal{K} : \Gamma \Uparrow L, \bot}\ [\bot] \quad \frac{\vdash \mathcal{K} : \Gamma \Uparrow L, A\{c/x\}}{\vdash \mathcal{K} : \Gamma \Uparrow L, \forall x.A}\ [\forall] \quad \frac{\vdash \mathcal{K} +_l A : \Gamma \Uparrow L}{\vdash \mathcal{K} : \Gamma \Uparrow L, ?^l A}\ [?^l]$$

$$\frac{\vdash \mathcal{K} : \Gamma \Downarrow A_i}{\vdash \mathcal{K} : \Gamma \Downarrow A_1 \oplus A_2}\ [\oplus_i] \quad \frac{\vdash \mathcal{K}_1 : \Gamma \Downarrow A \quad \vdash \mathcal{K}_2 : \Delta \Downarrow B}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \Gamma, \Delta \Downarrow A \otimes B}\ [\otimes, \text{given } (\mathcal{K}_1 = \mathcal{K}_2)|_{\mathcal{U}}]$$

$$\frac{}{\vdash \mathcal{K} : \cdot \Downarrow 1}\ [1, \text{given } \mathcal{K}[\mathcal{I} \setminus \mathcal{U}] = \emptyset] \quad \frac{\vdash \mathcal{K} : \Gamma \Downarrow A\{t/x\}}{\vdash \mathcal{K} : \Gamma \Downarrow \exists x.A}\ [\exists]$$

$$\frac{\vdash \mathcal{K} \leq_l : \cdot \Uparrow A}{\vdash \mathcal{K} : \cdot \Downarrow !^l A}\ [!^l, \text{given } \mathcal{K}[\{x \mid l \not\preceq x \wedge x \notin \mathcal{U}\}] = \emptyset]$$

$$\frac{}{\vdash \mathcal{K} : \Gamma \Downarrow A_t^{\perp}}\ [\text{I, given } A_t \in (\Gamma \cup \mathcal{K}[\mathcal{I}) \text{ and } (\Gamma \cup \mathcal{K}[\mathcal{I} \setminus \mathcal{U}]) \subseteq \{A_t\}]$$

$$\frac{\vdash \mathcal{K} +_l P : \Gamma \Downarrow P}{\vdash \mathcal{K} +_l P : \Gamma \Uparrow \cdot}\ [D_l, \text{given } l \in \mathcal{U}] \quad \frac{\vdash \mathcal{K} : \Gamma \Downarrow P}{\vdash \mathcal{K} +_l P : \Gamma \Uparrow \cdot}\ [D_l, \text{given } l \notin \mathcal{U}]$$

$$\frac{\vdash \mathcal{K} : \Gamma \Downarrow P}{\vdash \mathcal{K} : \Gamma, P \Uparrow \cdot}\ [D_1] \quad \frac{\vdash \mathcal{K} : \Gamma \Uparrow N}{\vdash \mathcal{K} : \Gamma \Downarrow N}\ [R \Downarrow] \quad \frac{\vdash \mathcal{K} : \Gamma, S \Uparrow L}{\vdash \mathcal{K} : \Gamma \Uparrow L, S}\ [R \Uparrow]$$

Fig. 1: Focused linear logic system with subexponentials. We assume that all atoms are classified as negative polarity formulas and their negations as positive polarity formulas. Here, L is a list of formulas, Γ is a multi-set of formulas and positive literals, A_t is an atomic formula, P is a non-negative literal, S is a positive literal or formula and N is a negative formula.

- $(\mathcal{K}_1 \otimes \mathcal{K}_2)[i] = \begin{cases} \mathcal{K}_1[i] \cup \mathcal{K}_2[i] & \text{if } i \notin \mathcal{U} \\ \mathcal{K}_1[i] & \text{if } i \in \mathcal{U} \end{cases}$
- $\mathcal{K}[\mathcal{S}] = \bigcup\{\mathcal{K}[i] \mid i \in \mathcal{S}\}$
- $(\mathcal{K} +_l A)[i] = \begin{cases} \mathcal{K}[i] \cup \{A\} & \text{if } i = l \\ \mathcal{K}[i] & \text{otherwise} \end{cases}$
- $\mathcal{K} \leq_i [l] = \begin{cases} \mathcal{K}[l] & \text{if } i \leq l \\ \emptyset & \text{if } i \not\leq l \end{cases}$
- $(\mathcal{K}_1 \star \mathcal{K}_2)\mid_{\mathcal{S}}$ is true if and only if $(\mathcal{K}_1[j] \star \mathcal{K}_2[j])$

Fig. 2: Specification of operations on contexts. Here, $i \in I$, $j \in \mathcal{S}$, $\mathcal{S} \subseteq I$, and the binary connective $\star \in \{=, \subset, \subseteq\}$.

tively, formulas appearing on the left and on the right of sequents. For example, the formulas appearing in the sequent $B_1, \ldots, B_n \vdash C_1, \ldots, C_m$ are specified by the meta-level atoms: $\lfloor B_1 \rfloor, \cdots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \cdots, \lceil C_m \rceil$.

Given such a collection of meta-level atoms, it remains to decide where exactly these atoms are going to appear in the meta-level sequents. When using linear logic without subexponentials, the number of possibilities is quite limited. As the sequents of linear logic without subexponentials ($\vdash \Theta : \Gamma$) have only two contexts, namely an unbounded context (Θ) (which is treated as a set of formulas) and a bounded context (Γ) (which is treated as a multiset of formulas), there are only two options: the meta-level formula either belongs to one context or to the other. The use of subexponentials opens, on the other hand, a wider range of possibilities, as there is one context for *each* subexponential index. For instance, we can encode the object-level sequent above by using two subexponentials: l whose context stores $\lfloor\cdot\rfloor$ formulas and r whose context stores $\lceil\cdot\rceil$ formulas. The meta-level encoding of an object-level sequent would in this case have the following form[4] $\vdash \mathcal{L} :_\infty \lfloor B_1\rfloor, \cdots, \lfloor B_n\rfloor :_l \lceil C_1\rceil, \cdots, \lceil C_m\rceil :_r \cdot \Uparrow \cdot$. Moreover, if needed, one could further refine such specification and partition meta-level atoms in more contexts by using more subexponentials. For instance, the focused sequent of focused proof systems, such as LJQ^*, has an extra context, called *stoup*, where the focused formula is. To specify such a sequent, we use an additional subexponential index f, whose context contains the focused formula. As we show in the next subsection, when we describe how inference rules are specified, this refinement of linear logic sequents enables the specification of a number of structural properties of proof systems in an elegant fashion.

Moreover, in *SELLF*, subexponential contexts can be configured so to behave as sets or multisets. For instance, if we use the subexponentials signature $\langle\{l, r, \infty\}, \preceq, \{l, \infty\}\rangle$, with some preorder $\preceq$, the contexts for l and ∞ are treated as sets, while the context for r is treated as a multiset. Such situation would be useful for any proof system where the right-hand-side of its sequent behaves as a *multiset* of formulas and the left-hand-side behaves as a *set* of formulas, *e.g.*, the system LJ for intuitionistic logic. We could, alternatively, specify the contexts for both l and r to behave as multisets. In this case, l and r are bounded subexponentials. Such a specification is used when both sides of the object-level sequent behave as multisets, such as for the system $G1m$ [29] for minimal logic, which has explicit weakening and contraction rules.

3.2 Encoding Inference Rules

Inference rules of a system are specified using *monopoles* and *bipoles* [18]. These concepts are generalized next.

Definition 3.1
A *monopole* formula is a $SELLF$ formula that is built up from atoms and occurrences of the negative connectives, with the restriction that, for any label t, $?^t$ has atomic scope and that all atomic formulas, A, are necessarily under the scope of a subexponential question-mark, $?^t A$. A *bipole* is a formula built from monopoles and negated atoms using only positive connectives, with the additional restriction that $!^s$, $s \in I$, can only be applied to a monopole. We shall also insist that a bipole is either a negated atom or has a top-level positive connective.

The last restriction on bipoles forces them to be different from monopoles: bipoles are always positive formulas. Using the linear logic distributive properties, monopoles are equivalent to formulas of the form

$$\forall x_1 \ldots \forall x_p[\&_{i=1,\ldots,n} \bindnasrepma_{j=1,\ldots,m_i} ?^{t_{i,j}} A_{i,j}],$$

where $A_{i,j}$ is an atomic formula and $t_{i,j} \in I$. Similarly, bipoles can be rewritten as formulas of the form

$$\exists x_1 \ldots \exists x_p[\oplus_{i=1,\ldots,n} \otimes_{j=1,\ldots,m_i} C_{i,j}],$$

where $C_{i,j}$ are either negated atoms, monopole formulas, or the result of applying $!^s$ to a monopole formula to some $s \in I$.

Throughout this paper, the following invariant holds: the linear context to the left of the $\Uparrow$ and $\Downarrow$ on *SELLF* sequents is empty[5]. This invariant derives from the focusing discipline

[4] $\mathcal{L}$ is a theory specifying the proof system's introduction rules, which will be explained later.

[5] That is, the context Γ in $\vdash \mathcal{K} : \Gamma \Uparrow \cdot$ and in $\vdash \mathcal{K} : \Gamma \Downarrow F$ is empty.

and from the definition of bipoles above, namely, from the fact that all atomic formulas are under the scope of a $?^t$. This is illustrated by the derivation below. In particular, according to the focusing discipline, a bipole is necessarily introduced by such a derivation containing a single alternation of phases. We call these derivations *bipole-derivations*.

$$\dfrac{\cdots \quad \dfrac{\cdots \quad \dfrac{\dfrac{\vdash \mathcal{K}'_i : \cdot \Uparrow \cdot}{\vdash \mathcal{K}_i <_s : \cdot \Uparrow \bindnasrepma_{j=1,\dots,m_i} ?^{t_{i,j}} A_{i,j}}\,[m_i \times (\bindnasrepma, ?^t)] \quad \cdots}{\dfrac{\vdash \mathcal{K}_i <_s : \cdot \Uparrow \forall x_1 \dots \forall x_p [\&_{i=1,\dots,n} \bindnasrepma_{j=1,\dots,m_i} ?^{t_{i,j}} A_{i,j}]}{\vdash \mathcal{K}_i : \cdot \Downarrow !^s \forall x_1 \dots \forall x_p [\&_{i=1,\dots,n} \bindnasrepma_{j=1,\dots,m_i} ?^{l_{i,j}} A_{i,j}]}\,[!^s]}\,[p \times \forall, n \times \&] \quad \cdots}{\dfrac{\vdash \mathcal{K}' : \cdot \Downarrow \exists x_1 \dots \exists x_t [\oplus_{i=1,\dots,k} \otimes_{j=1,\dots,q_i} C_{i,j}]}{\vdash \mathcal{K} : \cdot \Uparrow \cdot}}\,[t \times \exists, k \times \oplus, q_i \times \otimes]$$

Notice that the derivation above contains a single positive and a single negative trunk. Moreover, if the connective $!^s$ is not present, then the rule $!^s$ is replaced by the rule $R \Downarrow$.

It turns out that one can match exactly the shape of a bipole-derivation with the shape of the inference rule the bipole encodes. Consider, for example, the following bipole $F = \exists A \exists B.[\lfloor A \supset B \rfloor^\perp \otimes (!^l ?^r \lceil A \rceil \otimes ?^l \lfloor B \rfloor)]$ encoding the $\supset$ left-introduction rule for intuitionistic logic, assuming the signature $\langle \{l, r, \infty\}, \{l \prec \infty, r \prec \infty\}, \{l, \infty\} \rangle$. The only way to introduce F in *SELLF* is by using a bipole-derivation of the following form, where $F \in \Theta$:

$$\dfrac{\dfrac{\vdash \Theta \overset{\vdots}{\infty} \lfloor \Gamma \rfloor, \lfloor A \supset B \rfloor \overset{\vdots}{l} \lceil A \rceil \overset{\vdots}{r} \cdot \Uparrow \cdot \qquad \vdash \Theta \overset{\vdots}{\infty} \lfloor \Gamma \rfloor, \lfloor A \supset B \rfloor, \lfloor B \rfloor \overset{\vdots}{l} \lceil G \rceil \overset{\vdots}{r} \cdot \Uparrow \cdot}{\vdash \Theta \overset{\vdots}{\infty} \lfloor \Gamma \rfloor, \lfloor A \supset B \rfloor \overset{\vdots}{l} \lceil G \rceil \overset{\vdots}{r} \cdot \Downarrow F}}{\vdash \Theta \overset{\vdots}{\infty} \lfloor \Gamma \rfloor, \lfloor A \supset B \rfloor \overset{\vdots}{l} \lceil G \rceil \overset{\vdots}{r} \cdot \Uparrow \cdot}$$

The bipole-derivation above corresponds exactly to the left implication introduction rule for intuitionistic logic with premises $\Gamma, A \supset B \longrightarrow A$ and $\Gamma, A \supset B, B \longrightarrow G$, and conclusion $\Gamma, A \supset B \longrightarrow G$. Nigam and Miller in [22] classify this adequacy as *on the level of derivations*. Notice the role of $!^l$ in the derivation above. In order to introduce it, it must be the case that the context of subexponential r is empty. That is, the formula $\lceil G \rceil$ is necessarily moved to the right branch. All the proof systems that we encode in this paper (in Section 4) have this level of adequacy.

Subexponentials greatly increase the expressiveness of the framework allowing a number of structural properties of rules to be expressed. One can, *e.g.*, specify rules where (1) formulas in one or more contexts must be erased in the premise as well as rules that (2) require the presence of some formula in the context. We informally illustrate these applications of subexponentials.

For the first type of structural restriction, consider the following inference rule of the multi-conclusion system for intuitionistic logic:

$$\dfrac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow \Delta, A \supset B}\,[\supset_R]$$

Here, the set of formulas Δ has to be erased in the premise. This inference rule can be specified as the bipole $F = \exists A \exists B.\lceil A \supset B \rceil^\perp \otimes !^l(?^l \lfloor A \rfloor \bindnasrepma ?^r \lceil B \rceil)$, using the subexponential signature $\langle \{l, r, \infty\}, \{l \prec \infty, r \prec \infty\}, \{l, r, \infty\} \rangle$ where all contexts behave like sets. A bipole-derivation introducing this formula has necessarily the following shape, where $F \in \Theta$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Gamma\rfloor \mathbin{\dot{:}_{l}} \lceil\Delta, A\supset B\rceil \mathbin{\dot{:}_{r}} \cdot \Downarrow \lceil A\supset B\rceil^{\perp}}\,[I] \qquad \dfrac{\dfrac{\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Gamma, A\rfloor \mathbin{\dot{:}_{l}} \lceil B\rceil \mathbin{\dot{:}_{r}} \cdot \Uparrow}{\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Gamma\rfloor \mathbin{\dot{:}_{l}} \cdot \mathbin{\dot{:}_{r}} \cdot \Uparrow ?^{l}\lfloor A\rfloor \parr ?^{r}\lceil B\rceil)}\,[\parr, ?^{r}, ?^{l}]}{\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Gamma\rfloor \mathbin{\dot{:}_{l}} \lceil\Delta, A\supset B\rceil \mathbin{\dot{:}_{r}} \cdot \Downarrow !^{l}(?^{l}\lfloor A\rfloor \parr ?^{r}\lceil B\rceil)}\,[!^{l}]}{\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Gamma\rfloor \mathbin{\dot{:}_{l}} \lceil\Delta, A\supset B\rceil \mathbin{\dot{:}_{r}} \cdot \Downarrow \lceil A\supset B\rceil^{\perp} \otimes !^{l}(?^{l}\lfloor A\rfloor \parr ?^{r}\lceil B\rceil)}\,[\otimes]}{\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Gamma\rfloor \mathbin{\dot{:}_{l}} \lceil\Delta, A\supset B\rceil \mathbin{\dot{:}_{r}} \cdot \Uparrow}\,[D_{\infty}, 2\times\exists]
$$

Notice the role of the $!^{l}$ in the derivation above. It specifies that all formulas in the context of the subexponential r, *i.e.*, the formulas $\lceil\Delta, A\supset B\rceil$, should be weakened, hence corresponding exactly to the $\supset_R$ rule above.

In the example above, we showed how to specify systems where a single context should be erased. It is possible to generalize this idea to erasing any number of contexts: as before, this is done by specifying the pre-order $\preceq$ accordingly.

In some cases, however, we may also make use of *logical equivalences* and "*dummy*" indexes whose contexts will not store any formulas, but are just used to specify the structural restrictions of inference rules. For example, in the following rule of modal logic, the contexts Γ' and Δ' are both erased

$$
\dfrac{\Box\Gamma \vdash A, \diamond\Delta}{\Box\Gamma, \Gamma' \vdash \Box A, \diamond\Delta, \Delta'}\ [\Box_R]
$$

In order to specify this rule, we use the following set of subexponential indexes $\{l, r, \Box_l, \diamond_r, e, \infty\}$, where all indexes are unbounded. The contexts for l and r store formulas in the left and right-hand side, while the context for $\diamond_l$ and $\Box_r$ store formulas whose main connective is a diamond and box on the left and on the right-hand side, respectively. For instance, the sequent $\Box\Gamma, \Gamma', \diamond\Gamma'' \vdash \Box\Delta, \Delta', \diamond\Delta''$ is encoded as $\vdash \Theta \mathbin{\dot{:}_{\infty}} \lfloor\Box\Gamma\rfloor \mathbin{\dot{:}_{\Box_l}} \lfloor\Gamma', \diamond\Gamma''\rfloor \mathbin{\dot{:}_{l}} \lceil\Box\Delta, \Delta'\rceil \mathbin{\dot{:}_{r}} \lceil\diamond\Delta''\rceil \mathbin{\dot{:}_{\diamond_r}} \cdot \Uparrow \cdot$, where Θ is the theory specifying the inference rules of the system. The following clauses, classified as structural clauses (see Definition 3.2), specify the relation among object-logic formulas whose main connective is a $\Box$ and a $\diamond$ and the context of the indexes $\Box_l$ and $\diamond_r$.

$$
(\Box_S)\quad \lfloor\Box A\rfloor^{\perp} \otimes ?^{\Box_l}\lfloor\Box A\rfloor \quad \text{and} \quad (\diamond_S)\quad \lceil\diamond A\rceil^{\perp} \otimes ?^{\diamond_r}\lceil\diamond A\rceil
$$

From these clauses we obtain the equivalences[6] $\forall A.\lfloor\Box A\rfloor \equiv ?^{\Box_l}\lfloor\Box A\rfloor$ and $\forall A.\lceil\diamond A\rceil \equiv ?^{\diamond_r}\lceil\diamond A\rceil$. That is, any formula of the form $\lfloor\Box A\rfloor$ can be placed in the context of $\Box_l$ and any formula of the form $\lceil\diamond A\rceil$ to the context of $\diamond_r$. Furthermore, we specify e as follows: $e \prec \Box_l$, $e \prec \diamond_r$, and $e \prec \infty$ and e is unrelated to the remaining subexponentials. Hence, the connective $!^{e}$ can play a similar role for the specification of the rule $\Box_R$ as the $!^{l}$ in the specification of the $\supset_R$ rule above. In particular, to introduce $!^{e}$, all contexts but $\Box_l$, $\diamond_r$ and ∞ have to be erased. It is easy to check that this operation is exactly the one needed for specifying the modal logic rule above. In Section 4, we show this specification in detail.

In combination to the use of bounded subexponentials, whose contexts behave as multisets, subexponentials can also be used to check whether a formula is present in the sequent. These type of requirement also often appears in inference rules, such as the one below for intuitionistic lax logic [9]:

$$
\dfrac{F, \Gamma \longrightarrow \bigcirc G}{\bigcirc F, \Gamma \longrightarrow \bigcirc G}\ [\bigcirc_L]
$$

[6] $F \equiv G$ denotes the formula $(F \parr G^{\perp}) \otimes (F^{\perp} \parr G)$.

The connective $\bigcirc$ on the left can be introduced only if the main connective of the formula on the right is also a $\bigcirc$. To specify this rule, we use the following subexponentials indexes: $\{l, r, \circ_r, \infty\}$, where l and ∞ are unrestricted, while r and $\circ_r$ are restricted. Moreover, $r \prec \circ_r$, $\circ_r \prec l$, and $\circ_r, l \prec \infty$. Similarly as in the modal logic example above, a formula $\lceil H \rceil$ is stored in the context of the subexponential $\circ_r$ only if H's main connective is $\bigcirc$, *i.e.*, $H = \bigcirc H'$ for some H'. This is also accomplished by using an analogous logical equivalence, namely, $\forall A.\lceil \bigcirc A \rceil \equiv ?^{\circ_r}\lceil \bigcirc A \rceil$, which is obtained by using the clause ($\bigcirc_S$) in Figure 12. It is then easy to check that the formula $\exists F.\lfloor \bigcirc F \rfloor^{\perp} \otimes !^{\circ_r}\lfloor F \rfloor$ specifies the rule above. In particular, the $!^{\circ_r}$ is used to check whether the formula on the right has $\bigcirc$ as main connective: if this is the case, then some formula of the form $\lceil \bigcirc G \rceil$ will be in the context $\circ_r$, while the context for r will be empty. Notice, that this specification does not mention any side-formulas of the sequent, not even the formula appearing on the right-hand-side of the sequent. As we argue later, the use of such declarative specifications will help us reason about proof systems.

3.3 Canonical Proof System Theories

The definition below classifies clauses into three different categories, namely the identity rules (Cut and Init rules), introduction rules, and structural rules, following usual terminology in proof theory literature [29].

Definition 3.2

i. In its most general form, the clause specifying the *cut rule* has the form to the left, while the clause specifying the *initial rule* has the form to the right:

$$\mathit{Cut} = \exists A.!^{a}?^{b}\lfloor A \rfloor \otimes !^{c}?^{d}\lceil A \rceil \qquad \text{and} \qquad \mathit{Init} = \exists A.\lfloor A \rfloor^{\perp} \otimes \lceil A \rceil^{\perp}$$

where a, c are subexponentials that may or may not appear, depending on the structural restrictions imposed by the proof system.

ii. The *structural rules* are specified by clauses of the form below, where $i, j \in I$:

$$\exists A.[\lfloor A \rfloor^{\perp} \otimes (?^{i}\lfloor A \rfloor \parr \cdots \parr ?^{i}\lfloor A \rfloor)] \qquad \text{or} \qquad \exists A.[\lceil A \rceil^{\perp} \otimes (?^{j}\lceil A \rceil \parr \cdots \parr ?^{j}\lceil A \rceil)].$$

iii. Finally, an *introduction clause* is a closed bipole formula of the form

$$\exists x_1 \ldots \exists x_n[(q(\diamond(x_1, \ldots, x_n)))^{\perp} \otimes B]$$

where $\diamond$ is an object-level connective of arity n ($n \geq 0$) and $q \in \{\lfloor \cdot \rfloor, \lceil \cdot \rceil\}$. Furthermore, B does not contain negated atoms and an atom occurring in B is either of the form $p(x_i)$ or $p(x_i(y))$ where $p \in \{\lfloor \cdot \rfloor, \lceil \cdot \rceil\}$ and $1 \leq i \leq n$. In the first case, x_i has type *obj* while in the second case x_i has type $d \to \mathit{obj}$ and y is a variable (of type d) quantified (universally or existentially) in B (in particular, y is not in $\{x_1, \ldots, x_n\}$).

In the remainder of this paper, we restrict our discussion to the so called *canonical systems* [2].

Definition 3.3

A *canonical clause* is an introduction clause restricted so that, for every pair of atoms of the form $\lfloor T \rfloor$ and $\lceil S \rceil$ in a body, the head variable of T differs from the head variable of S. A *canonical proof system theory* is a set $\mathcal{X}$ of formulas such that (*i*) the *Init* and *Cut* clauses are members of $\mathcal{X}$; (*ii*) structural clauses may be members of $\mathcal{X}$; and (*iii*) all other clauses in $\mathcal{X}$ are canonical introduction clauses.

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, B \longrightarrow C}{\Gamma_1, \Gamma_2, A \supset B \longrightarrow C}\ [\supset L] \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ [\supset R] \qquad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ [\wedge L]$$

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \wedge B}\ [\wedge R] \qquad \frac{\Gamma, A\{t/x\} \longrightarrow C}{\Gamma, \forall x A \longrightarrow C}\ [\forall L] \qquad \frac{\Gamma \longrightarrow A\{c/x\}}{\Gamma \longrightarrow \forall x A}\ [\forall R]$$

$$\frac{\Gamma, A\{c/x\} \longrightarrow C}{\Gamma, \exists x A \longrightarrow C}\ [\exists L] \qquad \frac{\Gamma \longrightarrow A\{t/x\}}{\Gamma \longrightarrow \exists x A}\ [\exists R] \qquad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ [\vee L]$$

$$\frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \vee A_2}\ [\vee_i R] \qquad \frac{\Gamma \longrightarrow C}{\Gamma, A \longrightarrow C}\ [W_L] \qquad \frac{\Gamma, A, A \longrightarrow C}{\Gamma, A \longrightarrow C}\ [C_L]$$

$$\frac{}{A \longrightarrow A}\ [\text{Init}] \qquad \frac{\Gamma_1 \longrightarrow A \quad \Gamma_2, A \longrightarrow C}{\Gamma_1, \Gamma_2 \longrightarrow C}\ [\text{Cut}]$$

FIG. 3. The sequent calculus system *G1m* for minimal logic.

$(\supset_L)$	$\lfloor A \supset B \rfloor^\perp \otimes (!^l ?^r \lceil A \rceil \otimes ?^l \lfloor B \rfloor)$	$(\supset_R)$	$\lceil A \supset B \rceil^\perp \otimes !^l (?^l \lfloor A \rfloor \parr ?^r \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (?^l \lfloor A \rfloor \parr ?^l \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (!^l ?^r \lceil A \rceil \otimes !^l ?^r \lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (?^l \lfloor A \rfloor \,\&\, ?^l \lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (!^l ?^r \lceil A \rceil \oplus !^l ?^r \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes ?^l \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes !^l \forall x ?^r \lceil Bx \rceil$
$(\exists_L)$	$\lfloor \exists B \rfloor^\perp \otimes \forall x ?^l \lfloor Bx \rfloor$	$(\exists_R)$	$\lceil \exists B \rceil^\perp \otimes !^l ?^r \lceil Bx \rceil$
(Init)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	(Cut)	$!^l ?^r \lceil B \rceil \otimes ?^l \lfloor B \rfloor$
(C_L)	$\lfloor B \rfloor^\perp \otimes (?^l \lfloor B \rfloor \parr ?^l \lfloor B \rfloor)$	(W_L)	$\lfloor B \rfloor^\perp \otimes \perp$

FIG. 4. The theory, $\mathcal{L}_{G1m}$, for *G1m*.

4 Examples of Proof Systems encoded in *SELLF*

This section contains the specification of a number of proof systems that do not seem possible to be encoded in linear logic without the use of subexponentials or without mentioning side-formulas explicitly. In our specifications, we assume all free variables to be existentially quantified. Moreover, all the encodings below have the strongest level of adequacy, namely adequacy on the level of derivations [22].

4.1 G1m

The system *G1m* (Figure 3) for minimal logic contains explicit rules for weakening and contraction of formulas appearing on the left-hand-side of sequents. The encoding of this system illustrates how to use subexponentials to specify proof systems whose sequents contain two or more linear contexts. Here, in particular, both the left and the right-hand-side of *G1m* sequents are treated as multisets of formulas.

We specify *G1m* by using the following subexponential signature: $\langle \{\infty, l, r\}, \{r \prec l \prec \infty\}, \{\infty\} \rangle$. The subexponentials l and r do not allow neither contraction nor weakening. Their contexts will store, respectively, object-logic formulas appearing on the left and on the right of the sequent. Moreover, we use the theory $\mathcal{L}_{G1m}$, depicted in Figure 4, in order to specify in *SELLF* the *G1m*'s introduction rules. This theory is, on the other hand, stored in the context of ∞. Thus, a *G1m* sequent of the form $\Gamma \vdash C$ is encoded as the *SELLF* sequent $\vdash \mathcal{L}_{G1m} \stackrel{\infty}{:} \lfloor \Gamma \rfloor \stackrel{l}{:} \lceil C \rceil \stackrel{r}{:} \cdot \Uparrow \cdot$.

Each clause in $\mathcal{L}_{G1m}$ corresponds to one introduction rule of *G1m*. To obtain such strong correspondence, we need to capture precisely the structural restrictions in the system. In particular, the use of the $!^l$ in the clauses ($\supset_L$), specifying the rule $\supset_L$, and (Cut), specifying Cut rules, is necessary. It forces that the side-formula, C, appearing in the right-hand-side of their conclusion is moved to the correct premise. This is illustrated by the following derivation:

$$\dfrac{\dfrac{\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma_1\rfloor \overset{:}{l} \lceil A\rceil \overset{:}{r} \cdot \Uparrow}{\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma_1\rfloor \overset{:}{l} \cdot \overset{:}{r} \cdot \Downarrow !^l?^r\lceil A\rceil}\;[!^l, ?^r] \qquad \dfrac{\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma_2, A\rfloor \overset{:}{l} \lceil C\rceil \overset{:}{r} \cdot \Uparrow}{\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma_2\rfloor \overset{:}{l} \lceil C\rceil \overset{:}{r} \cdot \Downarrow ?^l\lfloor A\rfloor}\;[R\Downarrow, ?^l]}{\dfrac{\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma_1,\Gamma_2\rfloor \overset{:}{l} \lceil C\rceil \overset{:}{r} \cdot \Downarrow !^l?^r\lceil A\rceil \otimes ?^l\lfloor A\rfloor}{\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma_1,\Gamma_2\rfloor \overset{:}{l} \lceil C\rceil \overset{:}{r} \cdot \Uparrow}\;[D_\infty, \exists]}\;[\otimes]$$

When introducing the tensor, the formula $\lceil C\rceil$ cannot go to the left branch because, in that case, the r context would not be empty and therefore the $!^l$ could not be introduced. Hence, the only way to introduce the formula (Cut) in $\mathcal{L}_{G1m}$ is with a derivation as the one above.

In contrast, it is not possible to encode $G1m$ in linear logic (without subexponentials) with such a strong correspondence. The sequents of the dyadic version of linear logic [1] have only two contexts, one for the unbounded formulas and another for the linear formulas. Hence, in linear logic, all linear meta-level atoms would appear in the same context illustrated by the sequent $\vdash \Theta : \lfloor\Gamma\rfloor, \lceil C\rceil$. Furthermore, using the linear logic ! enforces that not only $\lceil C\rceil$, but *all* linear formulas in this sequent, namely $\lfloor\Gamma\rfloor$ and $\lceil C\rceil$, are moved to a different branch. Therefore, one cannot capture, as done by using the subexponential bang $!^l$, that only $\lceil C\rceil$ is necessarily moved to a different branch as specified in the $G1m$ rules $\supset_L$ and Cut.

Finally, as the derivation above illustrates, the $!^l$s appearing in the specification of $G1m$'s introduction rules specify the structural restriction that $G1m$'s sequents contain exactly one formula on their right-hand-side. This allows us to specify these introduction rules without explicitly mentioning any side-formulas in the sequent, such as, the formula C in the Cut rule. As we show in Section 5, the use of such declarative specifications allow for simple proofs about the object-level systems, such as the proof that it admits cut-elimination.

Repeating this exercise for each inference rule, we establish the following adequacy result.

Proposition 4.1

Let $\Gamma \cup \{C\}$ be a set of object logic formulas, and let the subexponentials, l and r, be specified by the signature $\langle\{\infty, l, r\}, \{r \prec l \prec \infty\}, \{\infty\}\rangle$. Then the sequent $\vdash \mathcal{L}_{G1m} \overset{:}{\infty} \lfloor\Gamma\rfloor \overset{:}{l} \lceil C\rceil \overset{:}{r} \cdot \Uparrow$ is provable in $SELLF$ if and only if the sequent $\Gamma \longrightarrow C$ is provable in $G1m$.

4.2 mLJ

We now encode in $SELLF$ the multi-conclusion sequent calculus *mLJ* for intuitionistic logic depicted in Figure 5. Its encoding illustrates the use of subexponentials to specify rules requiring some formulas to be weakened. In particular, the *mLJ*'s rules $\supset_R$ and $\forall_R$ require that the formulas Δ appearing in their conclusions to be weakened in their premises.

Formally, the theory $\mathcal{L}_{mlj}$ is formed by the clauses shown in Figure 6 This theory specifies *mLJ*'s rules by using the subexponential signature $\langle\{\infty, l, r\}; \{l \prec \infty, r \prec \infty\}; \{\infty, l, r\}\rangle$. As before with the encoding of $G1m$, we make use of two subexponentials l and r to store, respectively, meta-level atoms $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, but now we allow both contraction and weakening to these subexponential indexes. As described in Section 3.2, the use of $!^l$ in the clauses ($\supset_R$) and ($\forall_R$) specifies that the formulas in the context r should be necessarily weakened. This is

$$\frac{\Gamma, A \supset B \longrightarrow A, \Delta \quad \Gamma, A \supset B, B \longrightarrow \Delta}{\Gamma, A \supset B \longrightarrow \Delta}\ [\supset_L] \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B, \Delta}\ [\supset_R]$$

$$\frac{\Gamma, A \wedge B, A, B \longrightarrow \Delta}{\Gamma, A \wedge B \longrightarrow \Delta}\ [\wedge_L] \qquad \frac{\Gamma \longrightarrow A \wedge B, A, \Delta \quad \Gamma \longrightarrow A \wedge B, B, \Delta}{\Gamma \longrightarrow A \wedge B, \Delta}\ [\wedge_R]$$

$$\frac{\Gamma, A \vee B, A, \longrightarrow \Delta \quad \Gamma, A \vee B, B \longrightarrow \Delta}{\Gamma, A \vee B \longrightarrow \Delta}\ [\vee_L] \qquad \frac{\Gamma \longrightarrow A \vee B, A, B, \Delta}{\Gamma \longrightarrow A \vee B, \Delta}\ [\vee_R]$$

$$\frac{\Gamma, \forall x A, A\{t/x\} \longrightarrow \Delta}{\Gamma, \forall x A \longrightarrow \Delta}\ [\forall_L] \qquad \frac{\Gamma \longrightarrow A\{c/x\}}{\Gamma \longrightarrow \Delta, \forall x A}\ [\forall_R]$$

$$\frac{\Gamma, \exists x A, A\{c/x\} \longrightarrow \Delta}{\Gamma, \exists x A \longrightarrow \Delta}\ [\exists_L] \qquad \frac{\Gamma \longrightarrow \Delta, \exists x A, A\{t/x\}}{\Gamma \longrightarrow \Delta, \exists x A}\ [\exists_R]$$

$$\frac{}{\Gamma, A \longrightarrow A, \Delta}\ [\mathrm{Init}] \qquad \frac{\Gamma \longrightarrow B, \Delta \quad \Gamma, B \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}\ [\mathrm{Cut}] \qquad \frac{}{\Gamma, \bot \longrightarrow \Delta}\ [\bot_L]$$

FIG. 5. The multi-conclusion intuitionistic sequent calculus, *mLJ*, with additive rules.

$(\supset_L)$	$\lfloor A \supset B\rfloor^\perp \otimes (?^r\lceil A\rceil \otimes ?^l\lfloor B\rfloor)$	$(\supset_R)$	$\lceil A \supset B\rceil^\perp \otimes !^l(?^l\lfloor A\rfloor \parr ?^r\lceil B\rceil)$
$(\wedge_L)$	$\lfloor A \wedge B\rfloor^\perp \otimes (?^l\lfloor A\rfloor \parr ?^l\lfloor B\rfloor)$	$(\wedge_R)$	$\lceil A \wedge B\rceil^\perp \otimes (?^r\lceil A\rceil \otimes ?^r\lceil B\rceil)$
$(\vee_L)$	$\lfloor A \vee B\rfloor^\perp \otimes (?^l\lfloor A\rfloor \otimes ?^l\lfloor B\rfloor)$	$(\vee_R)$	$\lceil A \vee B\rceil^\perp \otimes (?^r\lceil A\rceil \parr ?^r\lceil B\rceil)$
$(\forall_L)$	$\lfloor \forall B\rfloor^\perp \otimes ?^l\lfloor Bx\rfloor$	$(\forall_R)$	$\lceil \forall B\rceil^\perp \otimes !^l\forall x ?^r\lceil Bx\rceil$
$(\exists_L)$	$\lfloor \exists B\rfloor^\perp \otimes \forall x ?^l\lfloor Bx\rfloor$	$(\exists_R)$	$\lceil \exists B\rceil^\perp \otimes ?^r\lceil Bx\rceil$
$(\bot_L)$	$\lfloor \bot\rfloor^\perp$		
(Init)	$\lfloor B\rfloor^\perp \otimes \lceil B\rceil^\perp$	(Cut)	$?^l\lfloor B\rfloor \otimes ?^r\lceil B\rceil$
(*Pos*)	$\lfloor B\rfloor^\perp \otimes ?^l\lfloor B\rfloor$	(*Neg*)	$\lceil B\rceil^\perp \otimes ?^r\lceil B\rceil$

FIG. 6. Theory $\mathcal{L}_{mlj}$ for the multi-conclusion intuitionistic logic system *mLJ*.

illustrated by the following derivation introducing the formula $(\forall_R)$ in $\mathcal{L}_{mlj}$:

$$\dfrac{\dfrac{}{\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \lceil\Delta, \forall x A\rceil \mathrel{\dot{r}} \cdot \Downarrow \lceil\forall x A\rceil^\perp}\ [I_R] \qquad \dfrac{\dfrac{\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \lceil Ac\rceil \mathrel{\dot{r}} \cdot \Uparrow}{\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \cdot \mathrel{\dot{r}} \cdot \Uparrow \forall x ?^r\lceil Ax\rceil}\ [\forall, ?^r]}{\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \lceil\Delta, \forall x A\rceil \mathrel{\dot{r}} \cdot \Downarrow !^l\forall x ?^r\lceil Ax\rceil}\ [!^l]}{\dfrac{\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \lceil\Delta, \forall x A\rceil \mathrel{\dot{r}} \cdot \Downarrow \lceil\forall A\rceil^\perp \otimes !^l\forall x ?^r\lceil Ax\rceil}{\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \lceil\Delta, \forall x A\rceil \mathrel{\dot{r}} \cdot \Uparrow}\ [D_\infty, \exists]}\ [\otimes]$$

Since $l \npreceq r$, all formulas in the context r should be weakened in the premise of the promotion rule. The derivation above also illustrates how one can specify fresh values with the use of the universal quantifier. As in *mLJ*, the eigenvariable c cannot appear in Δ nor Γ.

The following result is proved by induction on the height of focused proofs.

PROPOSITION 4.2
Let $\Gamma \cup \Delta$ be a set of object-logic formulas, and let the subexponentials l and r be specified by the signature $\langle\{\infty, l, r\}; \{l \prec \infty, r \prec \infty\}; \{\infty, l, r\}\rangle$. Then the sequent $\vdash \mathcal{L}_{mlj} \mathrel{\dot{\infty}} \lfloor\Gamma\rfloor \mathrel{\dot{l}} \lceil\Delta\rceil \mathrel{\dot{r}} \cdot \Uparrow$ is provable in *SELLF* if and only if the sequent $\Gamma \longrightarrow \Delta$ is provable in *mLJ*.

$$\frac{\Gamma, A \supset B \to A; \cdot \quad \Gamma, A \supset B, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}\ [\supset_L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \to A \supset B; \Delta}\ [\supset_R]$$

$$\frac{\Gamma, A \vee B, A \vdash \Delta \quad \Gamma, A \vee B, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ [\vee_L] \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \to A \vee B; \Delta}\ [\vee_R]$$

$$\frac{\Gamma, A \wedge B, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ [\wedge_L] \qquad \frac{\Gamma \to A; \Delta \quad \Gamma \to B; \Delta}{\Gamma \to A \wedge B; \Delta}\ [\wedge_R]$$

$$\frac{}{\Gamma, A \to A; \Delta}\ [\text{Init}] \qquad \frac{\Gamma \to C; \Delta}{\Gamma \vdash C, \Delta}\ [D] \qquad \frac{}{\Gamma, \bot \vdash \Delta}\ [\bot_L]$$

FIG. 7: The the cut-free fragment of the focused multi-conclusion system for intuitionistic logic - *LJQ**.

(Init)	$\lfloor A \rfloor^\perp \otimes \lceil A \rceil^\perp$	$(\bot_L)$	$\lfloor \bot \rfloor^\perp$
$(\supset_L)$	$\lfloor A \supset B \rfloor^\perp \otimes (!^l ?^f \lceil A \rceil \otimes !^r ?^l \lfloor B \rfloor)$	$(\supset_R)$	$\lceil A \supset B \rceil^\perp \otimes !^l (?^l \lfloor A \rfloor \parr ?^r \lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (!^r ?^l \lfloor A \rfloor \otimes !^r ?^l \lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes !^r (?^r \lceil A \rceil \parr ?^r \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes !^r (?^l \lfloor A \rfloor \parr ?^l \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (!^r ?^f \lceil A \rceil \otimes !^r ?^f \lceil B \rceil)$

FIG. 8. The theory $\mathcal{L}_{ljq}$ encoding the cut-free fragment of the system *LJQ**.

4.3 LJQ*

The systems in the previous sections always required two contexts. There are systems, however, that require more than two contexts to be specified, such as the focused multi-conclusion system for intuitionistic logic *LJQ** depicted in Figure 7. This system is a variant of the system proposed by Herbelin [13, page 78] and it was used by Dyckhoff & Lengrand in [8]. *LJQ** has two types of sequents: unfocused sequents of the form $\Gamma \vdash \Delta$ and focused sequents of the form $\Gamma \to A; \Delta$ where the formula A, in the *stoup*, is focused on. Proofs are restricted as follows: the logical right introduction rules introduce only focused sequents, while the left introduction rules introduce only unfocused sequents. In this Section, we encode only its cut-free fragment. Later in Section 5, we elaborate on the challenges of encoding its cut rules.

We use the theory $\mathcal{L}_{ljq}$ depicted in Figure 8 to specify the system *LJQ** in *SELLF* together with the signature $\langle \{f, l, r, \infty\}; \{r \prec l \prec \infty\}; \{l, r, \infty\} \rangle$. Besides the subexponential ∞, we make use of three subexponentials: the first two, l and r, are as before, used to encode, respectively, the left and the right-hand-side of object-logic sequents, while the third subexponential, f, is new and used to encode the stoup of object-logic focused sequents. A *LJQ** sequent of the form $\Gamma \vdash \Delta$ is encoded in *SELLF* as the sequent $\vdash \mathcal{L}_{ljq} \mathrel{\dot{\infty}} \lfloor \Gamma \rfloor \mathrel{\dot{l}} \lceil \Delta \rceil \mathrel{\dot{r}} \cdot \mathrel{\dot{f}} \cdot \Uparrow \cdot$, while a *LJQ** sequent of the form $\Gamma \to A; \Delta$ is encoded by the sequent $\vdash \mathcal{L}_{ljq} \mathrel{\dot{\infty}} \lfloor \Gamma \rfloor \mathrel{\dot{l}} \lceil \Delta \rceil \mathrel{\dot{r}} \lceil A \rceil \mathrel{\dot{f}} \cdot \Uparrow \cdot$.

Notice that, differently from the previous encoding, the subexponentials r and l are related in the pre-order and moreover contraction and weakening are not available only to f. As before, the restrictions to sequents imposed by the focusing discipline are encoded implicitly by the use of subexponentials. The specification is such that positive rules can only be applied to the focused formula and that negative rules can only be applied when the stoup is empty.

To illustrate the fact that negative rules are only applicable when the stoup is empty, consider the following derivation introducing the clause $(\wedge_L)$, where $\mathcal{K}$ is an abbreviation for the

$$\frac{A, B, A \wedge B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\,[\wedge_L] \qquad \frac{\Gamma \vdash \Delta, A \wedge B, A \quad \Gamma \vdash \Delta, A \wedge B, B}{\Gamma \vdash \Delta, A \wedge B}\,[\wedge_R]$$

$$\frac{\Gamma, A \Rightarrow B \vdash A, \Delta \quad \Gamma, A \Rightarrow B, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\,[\Rightarrow_L] \qquad \frac{\Gamma, A \vdash B, A \Rightarrow B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\,[\Rightarrow_R]$$

$$\frac{\Gamma, \Box A, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,[\Box_L] \qquad \frac{\Box\Gamma \vdash A, \Diamond\Delta}{\Box\Gamma, \Gamma' \vdash \Box A, \Diamond\Delta, \Delta'}\,[\Box_R]$$

$$\frac{\Box\Gamma, A \vdash \Diamond\Delta}{\Box\Gamma, \Gamma', \Diamond A \vdash \Diamond\Delta, \Delta'}\,[\Diamond_L] \qquad \frac{\Gamma \vdash \Delta, \Diamond A, A}{\Gamma \vdash \Delta, \Diamond A}\,[\Diamond_R]$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\,[\mathrm{Init}] \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,[\mathrm{Cut}]$$

FIG. 9. The additive version of the proof system for classical modal logic S4.

$(\wedge_L)$	$\lfloor A \wedge B\rfloor^\perp \otimes (?^l\lfloor A\rfloor \parr ?^l\lfloor B\rfloor)$	$(\wedge_R)$	$\lceil A \wedge B\rceil^\perp \otimes (?^r\lceil A\rceil \otimes ?^r\lceil B\rceil)$
$(\Rightarrow_L)$	$\lfloor A \Rightarrow B\rfloor^\perp \otimes (?^r\lceil A\rceil \otimes ?^l\lfloor B\rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B\rceil^\perp \otimes (?^l\lfloor A\rfloor \parr ?^r\lceil B\rceil)$
$(\Box_L)$	$\lfloor \Box A\rfloor^\perp \otimes ?^l\lfloor A\rfloor$	$(\Box_R)$	$\lceil \Box A\rceil^\perp \otimes !^e?^r\lceil A\rceil$
$(\Diamond_L)$	$\lfloor \Diamond A\rfloor^\perp \otimes !^e?^l\lfloor A\rfloor$	$(\Diamond_R)$	$\lceil \Diamond A\rceil^\perp \otimes ?^r\lceil A\rceil$
(Init)	$\lfloor A\rfloor^\perp \otimes \lceil A\rceil^\perp$	(Cut)	$?^l\lfloor A\rfloor \otimes ?^r\lceil A\rceil$
$(\Box_S)$	$\lfloor \Box A\rfloor^\perp \otimes ?^{\Box_L}\lfloor \Box A\rfloor$	$(\Diamond_S)$	$\lceil \Diamond A\rceil^\perp \otimes ?^{\Diamond_R}\lceil \Diamond A\rceil$

FIG. 10. Figure with the theory $\mathcal{L}_{S4}$ encoding the system S4

context $\mathcal{L}_{ljq} \overset{\cdot}{\infty} \lfloor\Gamma'\rfloor \overset{\cdot}{l} \lceil\Delta\rceil \overset{\cdot}{r} \cdot \overset{\cdot}{f} \cdot$, and Γ' is the set $\Gamma \cup \{A \wedge B\}$:

$$\dfrac{\dfrac{}{\vdash \mathcal{K} \Downarrow \lfloor A \wedge B\rfloor^\perp}\,[I_L] \quad \dfrac{\vdash \mathcal{L}_{ljq} \overset{\cdot}{\infty} \lfloor\Gamma', A, B\rfloor \overset{\cdot}{l} \lceil\Delta\rceil \overset{\cdot}{r} \cdot \overset{\cdot}{f} \cdot \Uparrow}{\vdash \mathcal{L}_{ljq} \overset{\cdot}{\infty} \lfloor\Gamma'\rfloor \overset{\cdot}{l} \lceil\Delta\rceil \overset{\cdot}{r} \cdot \overset{\cdot}{f} \cdot \Downarrow !^r(?^l\lfloor A\rfloor \parr ?^l\lfloor B\rfloor)}\,[!^r, \parr, 2 \times ?^l]}{\dfrac{\vdash \mathcal{L}_{ljq} \overset{\cdot}{\infty} \lfloor\Gamma'\rfloor \overset{\cdot}{l} \lceil\Delta\rceil \overset{\cdot}{r} \cdot \overset{\cdot}{f} \cdot \Downarrow \lfloor A \wedge B\rfloor^\perp \otimes !^r(?^l\lfloor A\rfloor \parr ?^l\lfloor B\rfloor)}{\vdash \mathcal{L}_{ljq} \overset{\cdot}{\infty} \lfloor\Gamma'\rfloor \overset{\cdot}{l} \lceil\Delta\rceil \overset{\cdot}{r} \cdot \overset{\cdot}{f} \cdot \Uparrow \cdot}\,[D_\infty, 2 \times \exists]}\,[\otimes]$$

Since $r \not\preceq f$, the context f must be empty in order to introduce the $!^r$ in the right branch. On the other hand, since $r \prec l$, the l context is left untouched in the premise of this derivation, thus specifying precisely the $\wedge_L$ introduction rule.

The following proposition can be proved by induction on the height of focused proofs.

PROPOSITION 4.3
Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object logic formulas, and let the subexponentials l, r and f be specified by the signature $\langle\{f, l, r, \infty\}\{r \prec l \prec \infty\}; \{l, r, \infty\}\rangle$. Then the sequent $\vdash \mathcal{L}_{ljq} \overset{\cdot}{l} \lfloor\Gamma\rfloor \overset{\cdot}{r} \lceil\Delta\rceil \overset{\cdot}{f} \cdot : \cdot \Uparrow$ is provable in *SELLF* if and only if the sequent $\Gamma \vdash \Delta$ is provable in LJQ^*.

4.4 Modal Logic S4

We encode next the proof system for classical modal logic S4 depicted in Figure 9. The encoding of this system illustrates the use of logical equivalences and "dummy" subexponentials to encode the structural properties of systems. In particular, the rules $\Box_R$ and $\Diamond_L$ are the interesting ones. In order to introduce a $\Box$ on the right, the formulas on the left whose main connective is not $\Box$ (Γ') and the formulas on the right whose main connective is not $\diamond$ (Δ') are weakened.

Consider the following subexponential signature and the theory $\mathcal{L}_{S4}$ depicted in Figure 10:

$$\langle\{l, r, \Box_L, \diamond_R, e, \infty\}, \{r \prec \diamond_R \prec \infty, l \prec \Box_L \prec \infty, e \prec \diamond_R, e \prec \Box_L\}, \{l, r, \Box_L, \diamond_R, e, \infty\}\rangle.$$

As with the other systems that we encoded, the context of the subexponential l and r will contain formulas of the form $\lfloor A \rfloor$ and $\lceil A \rceil$, respectively. However, the contexts of the subexponentials $\Box_L$ and $\diamond_R$ will contain formulas only formulas of the form $\lfloor \Box A \rfloor$ and $\lceil \diamond A \rceil$, respectively, that is, formulas containing object-logic formulas whose main connective is $\Box$ and $\diamond$. This is specified by from the following equivalences derived from the structural clauses ($\Box_S$) and ($\diamond_S$) in $\mathcal{L}_{S4}$:

$$\forall A.(\lfloor \Box A \rfloor \equiv ?^{\diamond_L}\lfloor \Box A \rfloor) \quad \text{and} \quad \forall A.(\lceil \diamond A \rceil \equiv ?^{\Box_R}\lceil \Box A \rceil).$$

Thus, a sequent in S4 of the form $\Box\Gamma, \Gamma', \diamond\Gamma'' \vdash \diamond\Delta, \Delta', \Box\Delta''$ is encoded in *SELLF* by the sequent $\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'' \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Uparrow \cdot$. Notice that the context of the index e is empty. It is a "dummy" index that is not used to mark formulas, but to specify the structural properties of rules. In particular, the connective $!^e$ can be used to erase the context of the subexponentials l and r, as illustrated by its introduction rule shown below:

$$\frac{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \cdot \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \cdot \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Uparrow F}{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'' \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Downarrow !^e F}\ [!^e]$$

As e is not related to the indexes l and r in the preorder $\preceq$, the contexts for l and r must be empty in the premise of the rule above, *i.e.*, the formulas in these contexts must be weakened. These are exactly the restrictions needed for encoding the rules $\diamond_L$ and $\Box_R$ in S4, specified by the clauses ($\diamond_L$) and ($\Box_R$) containing $!^e$. For instance, the bipole derivation introducing the formula ($\Box_R$) has necessarily the following shape:

$$\dfrac{\dfrac{}{\vdash \mathcal{K} \Downarrow \lceil \Box A \rceil^{\perp}}\ [I_R] \qquad \dfrac{\dfrac{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \cdot \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil A \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Uparrow \cdot}{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \cdot \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \cdot \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Uparrow ?^r\lceil A \rceil}}{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'', \Box A \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Downarrow !^e ?^r \lceil A \rceil}\ [!^e]}{\dfrac{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'', \Box A \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Downarrow \lceil \Box A \rceil^{\perp} \otimes !^e ?^r \lceil A \rceil}{\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'', \Box A \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Uparrow \cdot}\ [D_\infty, \exists]}\ [\otimes]$$

where $\mathcal{K}$ is $\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'', \Box A \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot$. As one can easily check, the derivation above corresponds exactly to S4's rule $\Box_R$.

The following proposition can be easily proved by induction on the height of focused proofs.

Proposition 4.4

Let $\Gamma \cup \Gamma' \cup \Gamma'' \cup \Delta \cup \Delta' \cup \Delta''$ be a set of object logic formulas, and let the subexponentials l, r, $\Box_L$, $\diamond_R$, e, and ∞ be specified by the signature

$$\langle\{l, r, \Box_L, \diamond_R, e, \infty\}, \{r \prec \diamond_R \prec \infty, l \prec \Box_L \prec \infty, e \prec \diamond_R, e \prec \Box_L\}, \{l, r, \Box_L, \diamond_R, e, \infty\}\rangle.$$

Then the sequent $\vdash \mathcal{L}_{S4} \,\dot{\infty}\, \lfloor \Box\Gamma \rfloor \,\dot{\Box}_L\, \lfloor \Gamma', \diamond\Gamma'' \rfloor \,\dot{l}\, \lceil \diamond\Delta \rceil \,\dot{\diamond}_R\, \lceil \Delta', \Box\Delta'' \rceil \,\dot{r}\, \cdot \,\dot{e}\, \cdot \Uparrow \cdot$ is provable in *SELLF* if and only if the sequent $\Box\Gamma, \Gamma', \diamond\Gamma'' \vdash \diamond\Delta, \Delta', \Box\Delta''$ is provable in S4.

$$\frac{\Gamma, A \wedge B, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ [\wedge L] \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ [\wedge R]$$

$$\frac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ [\vee L] \qquad \frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \vee A_2}\ [\vee R_i]$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ [\supset L] \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ [\supset R]$$

$$\frac{\Gamma, \bigcirc A, A \longrightarrow \bigcirc B}{\Gamma, \bigcirc A \longrightarrow \bigcirc B}\ [\bigcirc L] \qquad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow \bigcirc A}\ [\bigcirc R]$$

$$\frac{}{\Gamma, A \longrightarrow A}\ [\text{Init}] \qquad \frac{\Gamma \longrightarrow A \quad \Gamma, A \longrightarrow C}{\Gamma \longrightarrow C}\ [\text{Cut}]$$

FIG. 11. The additive version of the proof system for minimal lax logics – *Lax*.

$$\begin{array}{llll}
(\wedge_l) & \lfloor A \wedge B \rfloor^{\perp} \otimes (?^l \lfloor A \rfloor \bindnasrepma ?^l \lfloor B \rfloor) & (\wedge_r) & \lceil A \wedge B \rceil^{\perp} \otimes (!^l ?^r \lceil A \rceil \otimes !^l ?^r \lceil B \rceil) \\
(\vee_l) & \lfloor A \vee B \rfloor^{\perp} \otimes (?^l \lfloor A \rfloor \,\&\, ?^l \lfloor B \rfloor) & (\vee_r) & \lceil A \vee B \rceil^{\perp} \otimes (!^l ?^r \lceil A \rceil \oplus !^l ?^r \lceil B \rceil) \\
(\supset_l) & \lfloor A \supset B \rfloor^{\perp} \otimes (!^l ?^r \lceil A \rceil \otimes ?^l \lfloor B \rfloor) & (\supset_r) & \lceil A \supset B \rceil^{\perp} \otimes !^l (?^l \lfloor A \rfloor \bindnasrepma !^l ?^r \lceil B \rceil) \\
(\bigcirc_L) & \lfloor \bigcirc A \rfloor^{\perp} \otimes !^{\circ_r} ?^l \lfloor A \rfloor & (\bigcirc_R) & \lceil \bigcirc A \rceil^{\perp} \otimes !^l ?^r \lceil A \rceil \\
(\text{I}) & \lfloor A \rfloor^{\perp} \otimes \lceil A \rceil^{\perp} & (\text{Cut}) & ?^l \lfloor A \rfloor \otimes !^l ?^r \lceil A \rceil \\
(\bigcirc_S) & \lceil \bigcirc A \rceil^{\perp} \otimes ?^{\circ_r} \lceil \bigcirc A \rceil & &
\end{array}$$

FIG. 12. The theory $\mathcal{L}_{Lax}$ encoding the system *Lax*

As a final remark, it is also possible to encode the proof system for intuitionistic S4, which only allows for at most one formula to be at the right-hand-side of sequents. The encoding is similar to the the encoding above for classical logic with the difference that it contains extra subexponential bangs for specifying this restriction on sequents, similar to what was done in our encoding of *G1m*. Formally, the encoding is based on the following subexponential signature with two dummy subexponentials e_l and e_r, where the former behaves as the one used in the encoding of classical logic, while the latter additionally checks that the context to the right-hand-side of sequents is empty:

$$\langle \{l, r, \Box_L, \diamond_R, e_l, e_r, \infty\}, \{r \prec \diamond_R \prec \infty, l \prec \Box_L \prec \infty, e_l \prec \diamond_R, e_l \prec \Box_L, e_r \prec \Box_L\}, \{l, \Box_L, \infty\} \rangle.$$

For instance, the introduction rule $\Box_R$ shown below is specified by the clause $\exists A.[\lceil \Box A \rceil^{\perp} \otimes !^{e_r} ?^r \lceil A \rceil]$.

$$\frac{\Box\Gamma \longrightarrow A}{\Box\Gamma, \Gamma' \longrightarrow \Box A}$$

4.5 Lax Logic

Our last example is the encoding of the proof system for minimal *Lax* logic depicted in Figure 11. Its encoding illustrates the use of subexponentials to specify that a formula can only be introduced if a side-formula is present in the premise. An example of such a rule is the introduction rule for $\bigcirc$ on the left. To introduce it on the left, the main connective of the formula on the right-hand-side must also be a $\bigcirc$. As we detailed next, we use subexponentials to perform such a check, without mentioning the formula on the right-hand-side, as described at the end of Section 3.2.

Consider the following signature $\langle\{l, r, \circ_r, \infty\}; \{r \prec \circ_r \prec l \prec \infty\}; \{l, \infty\}\rangle$. Intuitively, we will interpret an object-logic sequent of the forms $\Gamma \longrightarrow H$ and $\Gamma \longrightarrow \bigcirc G$ as the meta-level sequents, respectively, $\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma\rfloor \stackrel{.}{:}_{l} \cdot \stackrel{.}{:}_{\circ_r} \lceil H\rceil \stackrel{.}{:}_{r} \cdot \Uparrow \cdot$ and $\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc G\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Uparrow \cdot$. That is, the context of the index l will contain all the formula on the left-hand-side, while the formula to the right-hand side will either be in the context of r or the context of $\circ_r$. However, only object-level formulas whose main connective is $\bigcirc$ can be in the context of $\circ_r$. The encoding of the proof system Lax is given in Figure 12. As in the specification of S4, this is accomplished by using the following equivalence derived from the structural clause ($\bigcirc_S$):

$$\forall A.\lceil\bigcirc A\rceil \equiv ?^{\circ_r}\lceil\bigcirc A\rceil.$$

That is, one can move whenever needed a meta-level formula $\lceil\bigcirc A\rceil$ to the context of $\circ_r$.

In the specification $\mathcal{L}_{Lax}$, the clause ($\bigcirc_L$) is the most interesting one specifying the corresponding rule of the proof system. The $!^{\circ_r}$ specifies that the context of the the restriction that the formula on the right must be marked with a $\bigcirc$. This is illustrated by the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma, \bigcirc A\rfloor \stackrel{.}{:}_{l} \cdot \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Downarrow \lfloor\bigcirc A\rfloor^{\perp}}\,[I] \qquad \dfrac{\dfrac{\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma, \bigcirc A, A\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc B\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Uparrow \cdot}{\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma, \bigcirc A\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc B\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Uparrow ?^{l}\lfloor A\rfloor}\,[?^{l}]}{\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma, \bigcirc A\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc B\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Downarrow !^{\circ_r}?^{l}\lfloor A\rfloor}\,[!^{\circ_r}]}{\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma, \bigcirc A\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc B\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Downarrow \lfloor\bigcirc A\rfloor^{\perp} \otimes !^{\circ_r}?^{l}\lfloor A\rfloor}\,[\otimes]}{\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma, \bigcirc A\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc B\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Uparrow \cdot}\,[D_{\infty}, \exists]$$

Notice that due to the $!^{\circ_r}$, the context of r must be empty. That is, the formula $\lceil\bigcirc B\rceil$ must be in the context of $\circ_r$, or in other words the main connective of the object-logic formula to the right-hand-side is necessarily a $\bigcirc$.

Notice as well that since $r \prec \circ_r$, the clause ($\bigcirc_R$) is admissible in the theory. That is, a formula can move from the context of $\circ_r$ to the context of r. With respect to the proof system Lax this formula specifies exactly the rule $\bigcirc_R$, introducing the connective $\bigcirc$ on the right. Therefore, in order to obtain a stronger level of adequacy, namely on the level of derivations [22], we mention it explicitly in the encoding.

The following proposition is proved by induction on the height of derivations.

Proposition 4.5
Let $\Gamma \cup \{C\}$ be a set of object logic formulas, and let the subexponentials l,r and $\circ_r$ be specified by the signature $\langle\{l, r, \circ_r, \infty\}; \{r \prec \circ_r \prec l \prec \infty\}; \{l, \infty\}\rangle$. Then the sequent $\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma\rfloor \stackrel{.}{:}_{l} \cdot \stackrel{.}{:}_{\circ_r} \lceil C\rceil \stackrel{.}{:}_{r} \cdot \Uparrow \cdot$ is provable in *SELLF* if and only if the sequent $\Gamma \longrightarrow C$ is provable in Lax and the sequent $\vdash \mathcal{L}_{Lax} \stackrel{.}{:}_{\infty} \lfloor\Gamma\rfloor \stackrel{.}{:}_{l} \lceil\bigcirc C\rceil \stackrel{.}{:}_{\circ_r} \cdot \stackrel{.}{:}_{r} \cdot \Uparrow \cdot$ is provable in *SELLF* if and only if the sequent $\Gamma \longrightarrow \bigcirc C$ is provable in Lax.

5 Reasoning about Sequent Calculus

This section presents general and effective criteria for checking whether a proof system encoded in *SELLF* has important proof theoretic properties, namely, cut-elimination, invertibility of rules, and the completeness of atomic identity rules. Instead of proving each one of these properties from scratch, we just need to check whether the specification of a proof system satisfies the corresponding criteria. Moreover, we show that checking such criteria can be easily automated.

5.1 Cut-elimination for cut-coherent systems

The rule *Cut* is often presented as the rule below

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [Cut]$$

where $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ may be sets or multisets of formulas. The formula A is called the cut-formula. A proof system is said to have the cut-elimination property when the cut rule is admissible on this system, *i.e.*, every proof that uses cuts can be transformed into a cut-free proof. There at least two important consequences of the cut-elimination theorem, namely the *sub-formula property* and the *consistency* of the proof system. Cut-elimination was first proved by Gentzen [10] for proof systems for classical (LK) and intuitionistic logic (LJ). Gentzen's proof strategy has been re-used to prove the cut-elimination of a number of proof systems. The proof is quite elaborated and it involves a number of cases, thus being exhaustive and error prone. The strategy can be summarized by the following steps:

1. (Reduction to Principal Cuts) Transforming a proof with cuts into a proof with *principal cuts*, that is, a cut whose premises are derived by introducing the cut-formula itself. This is normally shown by permuting inference rules, *e.g.*, permuting the cut-rule over other introduction rules.
2. (Reduction to Atomic Cuts) Transforming a proof with principal cuts into a proof with atomic cuts. This is normally shown by reducing a cut with a complex cut-formula into (possible many) cuts with simpler cut-formulas.
3. (Elimination of Atomic Cuts) Transforming a proof with atomic cuts into a cut-free proof. This is normally shown by permuting atomic cuts over other introduction rules until it reaches the leaves and it is erased.

We provide a criteria for each one of the steps above. The step two is not problematic. In particular, a criteria for reducing principal cuts to atomic cuts was given by Pimentel and Miller in [18] when encoding systems in linear logic. This criteria easily extends to the use of *SELLF* (see Definition 5.6 and Theorem 5.8).

While for specifications in linear logic steps one and three did not cause any problems [18], for specifications in *SELLF* they do not work as smoothly. For the step three of eliminating atomic cuts, however, we could still find a simple criteria for when this step can be performed (see Definition 5.9 and Theorem 5.10). But determining criteria for when it is possible to transform arbitrary cuts into principal cuts (step one) turned out to be a real challenge. And it should be, since *SELLF* allows for much more complicated proof systems to be encoded, such as *mLJ* and *LJQ**, with the highest level of adequacy. There are at least three possible strategies or reductions one can use to perform this transformation:

- (Permute Cut Rules Upwards) As done by Gentzen, one can try to permute cuts over other introduction rules. The following is an example of such a transformation in $G1m$:

$$\dfrac{\Gamma \longrightarrow A \qquad \dfrac{\Gamma', A, F \longrightarrow G}{\Gamma', A \longrightarrow F \supset G}\ [\supset_R]}{\Gamma, \Gamma' \longrightarrow F \supset G}\ [\text{Cut}] \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \longrightarrow A \qquad \Gamma', A, F \longrightarrow G}{\Gamma, \Gamma', F \longrightarrow G}\ [\text{Cut}]}{\Gamma, \Gamma' \longrightarrow F \supset G}\ [\supset_R]$$

We identify a criteria for when such permutations are always possible (see Lemma 5.2).

- (Permute Introduction Rules Downwards) In some cases, it is not possible to permute the cut over an introduction rule. For instance, in the *mLJ* derivation to the left, it is not always possible to permute a cut over an $\supset_R$, because such a permutation would weaken the formulas in Δ, which may be needed in the proof of left premise of the cut rule.

$$\dfrac{\mathcal{S} \qquad \dfrac{\dfrac{\Gamma, A, B, F \longrightarrow G}{\Gamma, A \wedge B, F \longrightarrow G}\,[\wedge_L]}{\Gamma, A \wedge B \longrightarrow F \supset G, \Delta}\,[\supset_R]}{\Gamma \longrightarrow F \supset G, \Delta}\,[\text{Cut}] \quad \rightsquigarrow \quad \dfrac{\mathcal{S} \qquad \dfrac{\dfrac{\Gamma, A, B, F \longrightarrow G}{\Gamma, A, B \longrightarrow F \supset G, \Delta}\,[\supset_R]}{\Gamma, A \wedge B \longrightarrow F \supset G, \Delta}\,[\wedge_L]}{\Gamma \longrightarrow F \supset G, \Delta}\,[\text{Cut}]$$

  The strategy then is to permute downwards the rule introducing the cut-formula ($A \wedge B$) on the Cut's right premise, as illustrated by the derivation to the right. In some cases, however, the cut-formula might need to be introduced multiple times. For instance, in the following S4 derivation, the cut cannot permute upwards, but one can still introduce the cut-formula $\Box A$ on the right before introducing the formula $\Box F$. Only, in this case, the cut-formula is introduced twice, as illustrated by the derivation to the right.[7]

$$\dfrac{\mathcal{S} \qquad \dfrac{\dfrac{\Box\Gamma, \Box A, A \vdash \diamond\Delta, F}{\Box\Gamma, \Box A \vdash \diamond\Delta, F}\,[\Box_L]}{\Box\Gamma, \Gamma', \Box A \vdash \diamond\Delta, \Delta', \Box F}\,[\Box_R]}{\Box\Gamma, \Gamma' \vdash \diamond\Delta, \Delta', \Box F}\,[\text{Cut}] \quad \rightsquigarrow \quad \dfrac{\mathcal{S} \qquad \dfrac{\dfrac{\dfrac{\Box\Gamma, \Box A, A \vdash \diamond\Delta, F}{\Box\Gamma, \Box A \vdash \diamond\Delta, F}\,[\Box_L]}{\Box\Gamma, \Gamma', \Box A, A \vdash \diamond\Delta, \Delta', \Box F}\,[\Box_R]}{\Box\Gamma, \Gamma', \Box A \vdash \diamond\Delta, \Delta', \Box F}\,[\Box_L]}{\Box\Gamma, \Gamma' \vdash \diamond\Delta, \Delta', \Box F}\,[\text{Cut}]$$

  A similar case also appears in *mLJ*, *e.g.*, when the cut formula is $A \supset B$. We identify criteria for when an introduction rule can permute over another introduction rule (see Lemma 5.4), which handles the cases for *mLJ* and S4 illustrated above.

- (Transform one Cut into Another Cut) There are systems, such as LJQ^*, which have more than one cut rule. For instance, LJQ^* has eight different cut rules, three of them shown in Example 5.3. In these cases, for permuting a cut of one type over an introduction rule might involve transforming this cut into another type of cut. As these permutations involve more elaborated proof transformations, finding criteria that is not ad-hoc to one system is much more challenging (if not impossible) and we will not provide one here.

We start our discussion of cut-elimination on specified sequent systems by the permutability step (step one). For this purpose, we define the notion *permutation of clauses* and then establish criteria for permutation of cut and introduction clauses.

Definition 5.1
Given C_1 and C_2 clauses in a canonical proof system theory $\mathcal{X}$, we say that C_1 *permutes over* C_2 if, given an arbitrary focused proof π of a sequent S ending with a bipole derivation introducing C_2 followed by a bipole derivation introducing C_1, then there exists a focused proof π' of S ending with a bipole derivation introducing C_1 followed by a bipole derivation introducing C_2.

Lemma 5.2 (Criteria cut permutation)
Let $\mathcal{X}$ be a canonical proof system theory. A cut clause permutes over an introduction or structural clause $C \in \mathcal{X}$ if, for each $s, t \in I$ such that $!^s B$ appears in C and $?^t B'$ is a subformula of the monopole B, one of the following holds:[8]

---

[7] This problem of permuting cuts in the system S4 was emphasized by Stewart and Stouppa in [28] and the complete proof can be found in [16].

[8] Of course, if the subexponential $!^s$ is not present in C, then the restrictions on s don't apply.

1. $Cut = \exists A.!^a?^b\lfloor A\rfloor \otimes !^c?^d\lceil A\rceil$ and either:
   i. permutation by vacuously: $s \npreceq b$ and b is bounded; or $s \npreceq d$ and d is bounded;
   ii. permutation to the right: $s \preceq a, d$ and $c \preceq t$;
   iii. permutation to the left: $s \preceq b, c$ and $a \preceq t$;
2. $Cut = \exists A.!^a?^b\lfloor A\rfloor \otimes ?^d\lceil A\rceil$ and either:
   i. permutation by vacuously: $s \npreceq b$ and b is bounded; or $s \npreceq d$ and d is bounded;
   ii. permutation to the right: $s \preceq a, d$;
3. $Cut = \exists A.?^b\lfloor A\rfloor \otimes !^c?^d\lceil A\rceil$ and either:
   i. permutation by vacuously: $s \npreceq b$ and b is bounded; or $s \npreceq d$ and d is bounded;
   ii. permutation to the left: $s \preceq b, c$;
4. $Cut = \exists A.?^b\lfloor A\rfloor \otimes ?^d\lceil A\rceil$ and either:
   i. permutation by vacuously: $s \npreceq b$ and b is bounded; or $s \npreceq d$ and d is bounded;
   ii. permutation to the right or left: s is the least element of $\langle I, \preceq\rangle$.

Proof. Suppose that C is a formula of the shape $!^s?^tB$[9].

- Case $Cut = \exists A.!^a?^b\lfloor A\rfloor \otimes !^c?^d\lceil A\rceil$. Consider the proof:

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 \preceq_a +_b\lfloor A\rfloor : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_1 : \cdot \Downarrow !^a?^b\lfloor A\rfloor}\;[!^a,?^b] \quad \dfrac{\dfrac{\dfrac{\dfrac{\Xi_2'}{\vdash \mathcal{K}_2 \preceq_{c,s} +_d\lceil A\rceil +_t B : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_2 \preceq_c +_d\lceil A\rceil : \cdot \Downarrow !^s?^tB}\;[!^s,?^t]}{\vdash \mathcal{K}_2 \preceq_c +_d\lceil A\rceil : \cdot \Uparrow \cdot}\;[D_\infty]}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^c?^d\lceil A\rceil}\;[!^c,?^d]}{\dfrac{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow !^a?^b\lfloor A\rfloor \otimes !^c?^d\lceil A\rceil}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\;[D_\infty,\exists]}\;[\otimes]$$

  If $s \npreceq d$ and d is bounded this case will not happen and the permutation is by vacuously. Otherwise, if $s \preceq d$, $s \preceq a$ and $c \preceq t$, the proof above can be replaced by

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 \preceq_{s,a} +_b\lfloor A\rfloor : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_1 \preceq_s : \cdot \Downarrow !^a?^b\lfloor A\rfloor}\;[!^a,?^b] \quad \dfrac{\dfrac{\Xi_2'}{\vdash \mathcal{K}_2 \preceq_{s,c} +_tB +_d \lceil A\rceil : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_2 \preceq_s +_tB : \cdot \Downarrow !^c?^d\lceil A\rceil}\;[!^c,?^d]}{\dfrac{\dfrac{\dfrac{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 \preceq_s +_tB : \cdot \Downarrow !^a?^b\lfloor A\rfloor \otimes !^c?^d\lceil A\rceil}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 \preceq_s +_tB : \cdot \Uparrow \cdot}\;[D_\infty,\exists]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow !^s?^tB}\;[!^s,?^t]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\;[D_\infty]}\;[\otimes]$$

  Notice that, since $s \preceq a$, $\mathcal{K}_1 \preceq_{s,a} = \mathcal{K}_1 \preceq_a$. Hence, in this case, the permutation is to the right. The same reasoning can be done for the left premise.

[9] In fact, we should consider bipoles D containing subformulas of the form $!^sC$ with C a monopole, but we will present only the case where $D = !^s?^tB$ for readability purposes.

- Case $Cut = !^a?^b\lfloor B\rfloor \otimes ?^d\lceil B\rceil$. If $s \leq d$ and $s \leq a$, then the derivation

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 \leq_a +_b\lfloor A\rfloor : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_1 : \cdot \Downarrow !^a?^b\lfloor A\rfloor}\,[!^a, ?^b] \quad \dfrac{\dfrac{\dfrac{\dfrac{\Xi_2'}{\vdash \mathcal{K}_2 \leq_s +_d\lceil A\rceil +_t B : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_2 +_d \lceil A\rceil : \cdot \Downarrow !^s?^t B}\,[!^s, ?^t]}{\vdash \mathcal{K}_2 +_d \lceil A\rceil : \cdot \Uparrow \cdot}\,[D_\infty]}{\vdash \mathcal{K}_2 : \cdot \Downarrow ?^d\lceil A\rceil}\,[?^d]}{\dfrac{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow !^a?^b\lfloor A\rfloor \otimes ?^d\lceil A\rceil}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\,[D_\infty, \exists]}\,[\otimes]$$

can be replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 \leq_{s,a} +_b\lfloor A\rfloor : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_1 \leq_s : \cdot \Downarrow !^a?^b\lfloor A\rfloor}\,[!^a, ?^b] \quad \dfrac{\dfrac{\Xi_2'}{\vdash \mathcal{K}_2 \leq_s +_t B +_d \lceil A\rceil : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_2 \leq_s +_t B : \cdot \Downarrow ?^d\lceil A\rceil}\,[?^d]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 \leq_s +_t B : \cdot \Downarrow !^a?^b\lfloor A\rfloor \otimes ?^d\lceil A\rceil}\,[\otimes]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 \leq_s +_t B : \cdot \Uparrow \cdot}\,[D_\infty, \exists]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow !^s?^t B}\,[!^s, ?^t]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\,[D_\infty]$$

There is a very interesting observation in this case: the restrictions for permuting the cut clause over the left premise form a superset of the restrictions for the right premise. In fact, other the fact that s should be the least element of I, it should also be the case that $a \leq t$. That is, if the permutation is possible at all, it can be always done over the right premise. Finally, if $s \not\leq d$ and d is bonded, then focusing over $!^s?^t B$ is not possible at all (the same for the left premise).

- Case $Cut = ?^b\lfloor B\rfloor \otimes !^c?^d\lceil B\rceil$. Analogous to the last case.
- Case $Cut = ?^b\lfloor B\rfloor \otimes ?^d\lceil B\rceil$. If s is the least element of I, then the derivation

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 +_b \lfloor A\rfloor : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_1 : \cdot \Downarrow ?^b\lfloor A\rfloor}\,[?^b] \quad \dfrac{\dfrac{\dfrac{\dfrac{\Xi_2'}{\vdash \mathcal{K}_2 \leq_s +_d\lceil A\rceil +_t B : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_2 +_d \lceil A\rceil : \cdot \Downarrow !^s?^t B}\,[!^s, ?^t]}{\vdash \mathcal{K}_2 +_d \lceil A\rceil : \cdot \Uparrow \cdot}\,[D_\infty]}{\vdash \mathcal{K}_2 : \cdot \Downarrow ?^d\lceil A\rceil}\,[?^d]}{\dfrac{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow ?^b\lfloor A\rfloor \otimes ?^d\lceil A\rceil}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\,[D_\infty, \exists]}\,[\otimes]$$

can be replaced by[10]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 \leq_s +_b\lfloor A\rfloor : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_1 \leq_s : \cdot \Downarrow ?^b\lfloor A\rfloor}\,[?^b] \quad \dfrac{\dfrac{\Xi_2'}{\vdash \mathcal{K}_2 \leq_s +_t B +_d \lceil A\rceil : \cdot \Uparrow \cdot}}{\vdash \mathcal{K}_2 \leq_s +_t B : \cdot \Downarrow ?^d\lceil A\rceil}\,[?^d]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 \leq_s +_t B : \cdot \Downarrow ?^b\lfloor A\rfloor \otimes ?^d\lceil A\rceil}\,[\otimes]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 \leq_s +_t B : \cdot \Uparrow \cdot}\,[D_\infty, \exists]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow !^s?^t B}\,[!^s, ?^t]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\,[D_\infty]$$

[10] Observe that the permutation could be done also on the left premise.

■

Example 5.3

Note that, from the systems presented in Section 4, the cuts defined in systems $G1m$ and Lax permutes over any introduction or structural clause. This means that, for these systems, the classical argument of permuting cuts up the proof until getting principal cuts works fine.

mLJ's cut clause ($Cut_{mLJ} = \exists A.?^l\lfloor A\rfloor \otimes ?^r\lceil A\rceil$), on the other hand, does not permute over clauses ($\supset_R$) and ($\forall_R$), since $!^l$ is present in both clauses but neither r is bounded (while $l \npreceq r$) nor l is the least element in the signature $\langle\{\infty, l, r\}; \{l \preceq \infty, r \preceq \infty\}; \{\infty, l, r\}\rangle$. This captures well, at the meta-level, the fact that the cut rule does not permute over the rules ($\supset_R$) and ($\forall_R$) at the object-level.

In the same way, in S4, the cut clause $Cut_{S4} = \exists A.?^l\lfloor A\rfloor \otimes ?^r\lceil A\rceil$ does not permute over the clauses ($\Box_R$) and ($\Diamond_L$) since l, r are unbounded and e is not the least element of the signature

$$\langle\{l, r, \Box_L, \Diamond_R, e, \infty\}, \{r \preceq \Diamond_R \preceq \infty, l \preceq \Box_L \preceq \infty, e \preceq \Diamond_R, e \preceq \Box_L\}, \{l, r, \Box_R, \Diamond_R, \infty\}\rangle.$$

In *LJQ**, three cut rules are admissible[11]:

$$\frac{\Gamma_1 \to A; \Delta_1 \qquad A, \Gamma_2 \to B; \Delta_2}{\Gamma_1, \Gamma_2 \to B; \Delta_1, \Delta_2}\ [Cut_1] \qquad \frac{\Gamma_1 \to A; \Delta_1 \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [Cut_2]$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ [Cut_3]$$

The first rule cannot be encoded in *SELLF* using only bipoles with the signature presented in this paper. In fact, we would need to add "dummy" subexponentials for guaranteeing the presence of focused formulas on the context, more or less the same way done for the Lax logic. The other cut rules can be specified, respectively, by the clauses

$$(Cut_2) \quad !^r?^l\lfloor A\rfloor \otimes !^r?^f\lceil A\rceil \quad (Cut_3) \quad !^r?^l\lfloor A\rfloor \otimes !^r?^r\lceil A\rceil.$$

It is interesting to note that, in Cut_2, the permutation to the right is by vacuously with *every* clause in the system. And it should be so since, at the object level, the left premise of the Cut_2 rule has a focused right cut formula, which *must be* principal. Hence the cut rule cannot permute up in the object level, as the cut clause does not permute over any other clause of the system. For the permutation to the left, the conditions $s \preceq l$ and $s \preceq r$ and $r \preceq t$ for any clause of the form $!^sB(\cdots ?^tB')$ appearing in $\mathcal{L}_{ljq}$, implies that: $s = r$ and $t = r, l$. Hence the Cut_2 clause permutes to the left over $(\supset_L), (\vee_L), (\wedge_L)$ and $(\vee_R)$ and it does not permute over $(\supset_R)$ and $(\wedge_R)$. As it should be since, at the object level, the premises of the rule $(\wedge_R)$ are focused and, as already discussed for the system *mLJ*, the rule $(\supset_R)$ erases formulas of the premises, hence not permuting with the cut rule.

For the Cut_3 clause, the argument is similar to the one just presented and Cut_3 does not permute to the right or to the left with $(\supset_R)$ and $(\wedge_R)$, permuting over the other introduction clauses of the system. As said before, the cut-elimination process for *LJQ** is more involving, making use of exchange between cuts, and it will not be discussed in more details here.

The following lemma establishes criterias for checking when a clause permutes over another clause. It captures all the non-trivial permutations for the systems *mLJ* and S4, that is, all the cases that are not true by vacuously.

---

[11] In fact, there are five admissible cut rules in *LJQ**, but the other two are derived from those presented here. And it is also worthy to note that there are three non-admissible cut rules in *LJQ**.

Lemma 5.4 (Criteria introduction permutation)
Let $\mathcal{X}$ be a canonical system and $C_1, C_2 \in \mathcal{X}$ be introduction or structural clauses. Assume that all subexponentials are unbounded, *i.e.*, $I = \mathcal{U}$. Then C_1 permutes over C_2 if at least one of the following is satisfied:

1. If C_1 and C_2 have no occurrences of subexponential bangs;
2. If C_1 has at least one occurrence of a subexponential bang but C_2 has no occurrence of subexponential bang, then for all occurrences of a formula of the form $!^s B_1$ in C_1 and for all occurrences of $?^t$ in C_2, it is the case that at least one of the following is true:
   i. $s \preceq t$;
   ii. if $C_2 = \exists x_1 \ldots \exists x_n[(q(\diamond(x_1, \ldots, x_n)))^\perp \otimes B]$, where $q \in \{\lfloor\cdot\rfloor, \lceil\cdot\rceil\}$, then the following equivalence is derivable from the structural rules of $\mathcal{X}$, where $s \preceq v$: $q(\diamond(x_1, \ldots, x_n))) \equiv ?^v q(\diamond(x_1, \ldots, x_n)))$.
3. If C_2 has at least one occurrence of a subexponential bang but C_1 has no occurrence of subexponential bang, then, for all occurrences of a formula of the form $!^s B_1$ in C_2 and for all occurrences of $?^t$ in C_1, either:
   i. $s \not\preceq t$ (in this case, the clause C_1 is unnecessary and can be dropped);
   ii. s is the least element of I.
4. If both C_1 and C_2 have at least one occurrence of a subexponential bang, then for each $s_k, t_k \in I$, $k = \{1, 2\}$, such that $!^{s_k} B_k$ appears in C_k and $?^{t_k} B'_k$ is a subformula of the monopole B_k, at least one of the following is true:
   i. $s_2 \not\preceq t_1$ and $s_1 \preceq s_2$ (in this case, the clause C_1 is unnecessary and can be dropped);
   ii. s_2 is the least element of I and $s_1 \preceq t_2$.

Proof. The assumption that all subexponentials are unbounded eliminates any problems caused by the splitting of formulas in the context, such as the case of permuting a & over a $\otimes$. As all formulas in the context are unbounded, we do not need to split them. Hence, we only have to analyze the problems due to the subexponentials.

The case when C_1 and C_2 do not contain subexponential bangs is easy. We show only the second case, when C_1 has a subexponential bang, but C_2 does not. The remaining cases follow similarly. The following piece of derivation illustrates how the permutation is possible.

$$\dfrac{\cdots \qquad \dfrac{\cdots \qquad \dfrac{\cdots \quad \dfrac{\overset{\Xi}{\vdash \mathcal{K} \leq_s +_u B +_t A : \cdot \Uparrow \cdot}}{\vdash \mathcal{K} \leq_s +_u B : \cdot \Uparrow ?^t A}\,[?^t] \quad \cdots}{\dfrac{\dfrac{\vdash \mathcal{K} \leq_s +_u B : \cdot \Downarrow C_2}{\vdash \mathcal{K} \leq_s +_u B : \cdot \Uparrow}\,[D_\infty]}{\vdash \mathcal{K} \leq_s : \cdot \Uparrow ?^u B}\,[?^u]} \qquad \cdots}{\dfrac{\vdash \mathcal{K} \leq_s : \cdot \Uparrow B_1}{\vdash \mathcal{K} : \cdot \Downarrow !^s B_1}\,[!^s]} \qquad \cdots}{\dfrac{\vdash \mathcal{K} : \cdot \Downarrow C_1}{\vdash \mathcal{K} : \cdot \Uparrow}\,[D_\infty]}$$

If $s \preceq t$, we can obtain the proof below where with a decide rule on C_2 appearing at the bottom[12].

$$
\dfrac{\cdots \qquad \dfrac{\cdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Xi}{\vdash \mathcal{K} \leq_s +_t A +_u B : \cdot \Uparrow \cdot}}{\vdash \mathcal{K} \leq_s +_t A : \cdot \Uparrow ?^u B}\,[?^u] \quad \cdots}{\dfrac{\vdash \mathcal{K} \leq_s +_t A : \cdot \Uparrow B_1}{\vdash \mathcal{K} +_t A : \cdot \Downarrow !^s B_1}\,[!^s]} \qquad \cdots}{\dfrac{\vdash \mathcal{K} +_t A : \cdot \Downarrow C_1}{\vdash \mathcal{K} : \cdot \Uparrow ?^t A}\,[?^t]}\,[D_\infty] \qquad \cdots}{\dfrac{\vdash \mathcal{K} : \cdot \Downarrow C_2}{\vdash \mathcal{K} : \cdot \Uparrow}\,[D_\infty]}}
$$

For when $s \not\leq t$, then the introduction of $!^s$ will cause the weakening of the formula A, However, if $q(\diamond(x_1, \ldots, x_n))) \equiv ?^v q(\diamond(x_1, \ldots, x_n)))$, where $q(\diamond(x_1, \ldots, x_n)))$ is the formula used by C_2 and $s \leq v$, then there is a derivation where $q(\diamond(x_1, \ldots, x_n)))$ is not weakened by the introduction of $!^s$. Hence, it is possible to focus on C_2 again after focusing on C_1 and recover the formula A.

∎

Observe that this last lemma is much more involving than Lemma 5.2. In fact, the cut clause is a formula with no head, and what it roughly does is to split the context into two and add a left formula in one part and a right formula in the other. When permuting two introduction clauses, on the other hand, one has to be careful not erasing contexts that will be necessary for the application of the next clause. For instance, the head of the clause C_1 can be in a context that will be eventually erased by the clause C_2, hence the exchange cannot happen.

As said before, our main interest on permuting clauses is to be able to consider only object-level *principal cuts*. We will clarify better now this concept. Let $\mathcal{X}$ be a canonical system and Ξ be a *SELLF* proof of the sequent $\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow \cdot$ ending with an introduction of the *Cut* clause. The premise of that decide rule is the conclusion of an $[\exists]$ infer rule. Let A be the substitution term used to instantiate the existential quantifier. We say that this occurrence of the $[D_\infty]$ inference rule is an *object-level cut* with *cut formula* A. Suppose $A = \diamond(\bar{B})$ is a non-atomic object level formula with left and right introduction rules

$$\exists \bar{x}(\lfloor \diamond(\bar{x}) \rfloor^\perp \otimes B_l) \quad \text{and} \quad \exists \bar{x}(\lceil \diamond(\bar{x}) \rceil^\perp \otimes B_r)$$

We say that this introduction of the *Cut* clause is *principal* if Ξ has the form

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash \mathcal{K}_1 \leq_a +_b \lfloor \diamond(\bar{B}) \rfloor : \cdot \Downarrow B_l[\bar{B}/\bar{x}]}}{\vdash \mathcal{K}_1 \leq_a +_b \lfloor \diamond(\bar{B}) \rfloor : \cdot \Uparrow \cdot}\,[D_\infty, \exists, \otimes, I]}{\vdash \mathcal{K}_1 : \cdot \Downarrow !^a ?^b \lfloor \diamond(\bar{B}) \rfloor}\,[!^a, ?^b] \qquad \dfrac{\dfrac{\dfrac{\Xi_2}{\vdash \mathcal{K}_2 \leq_c +_d \lceil \diamond(\bar{B}) \rceil : \cdot \Downarrow B_r[\bar{B}/\bar{x}]}}{\vdash \mathcal{K}_2 \leq_c +_d \lceil \diamond(\bar{B}) \rceil : \cdot \Uparrow \cdot}\,[D_\infty, \exists, \otimes, I]}{\vdash \mathcal{K}_2 : \cdot \Downarrow !^c ?^d \lceil \diamond(\bar{B}) \rceil}\,[!^c, ?^d]}{\dfrac{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow !^a ?^b \lfloor \diamond(\bar{B}) \rfloor \otimes !^c ?^d \lceil \diamond(\bar{B}) \rceil}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Uparrow}\,[D_\infty, \exists]}\,[\otimes]
$$

Definition 5.5

Let $\mathcal{X}$ be a canonical proof system theory. We say that $\mathcal{X}$ is *cut-principal* if every proof Ξ of a sequent S of the form $\vdash \mathcal{K} : \Delta \Uparrow \cdot$, with $\mathcal{K}[\infty] = \mathcal{X}$, having an introduction of a *Cut*

[12] Since $s \leq t$, the introduction of $!^s$ does not cause the weakening of the formula A.

clause, can be transformed, using permutations over clauses, into a proof Ξ' of S where that introduction of the *Cut* clause is principal.

Hence, for example, the systems $G1m$ and *Lax* are cut-principal, since their cut clauses permutes over any other clause of the system. A straightforward case analysis shows that *mLJ* and S4 also have this property: when cuts cannot permute up, rules can permute down, making the cuts principal.

Once we can transform an introduction of a cut into a principal one, the proof of cut elimination for logical systems continues by showing how to transform a principal cut into cuts with "simpler" formulas. This transformation is often based on the fact that systems have "dual" introduction rules for each connective. In [18], Pimentel and Miller introduced the concept of cut-coherence for linear logic specifications that captures this notion of duality. We extend this definition to our setting with subexponentials.

Definition 5.6
Let $\mathcal{X}$ be a canonical proof system theory and $\diamond$ an object-level connective of arity $n \geq 0$. Furthermore, let the formulas

$$\exists \bar{x}(\lfloor \diamond(\bar{x}) \rfloor^{\perp} \otimes B_l) \quad \text{and} \quad \exists \bar{x}(\lceil \diamond(\bar{x}) \rceil^{\perp} \otimes B_r)$$

be the left and right introduction rules for $\diamond$, where the free variables of B_l and B_r are in the list of variables $\bar{x}$. The object-level connective $\diamond$ has *cut-coherent introduction rules* if the sequent $\vdash \mathcal{K}_\infty : \cdot \Uparrow \forall \bar{x}(B_l^{\perp} \parr B_r^{\perp})$ is provable in *SELLF*, where $\mathcal{K}_\infty[\infty] = \{Cut\}$, $\{Cut\}$ is the set of all cut clauses in $\mathcal{X}$ and $\mathcal{K}_\infty[i] = \emptyset$ for any other $i \in I$. A canonical proof system theory is called *cut-coherent* if all object-level connectives have cut-coherent introduction rules.

Example 5.7
The cut-coherence of the $G1m$ specification is established by proving the following sequents.

$$
\begin{array}{ll}
(\supset) & \vdash Cut_{G1m} \stackrel{\vdots}{\infty} \cdot \stackrel{\vdots}{l} \cdot \stackrel{\vdots}{r} \cdot \stackrel{\vdots}{f} \cdot \Uparrow ?^l !^r \lceil A \rceil^{\perp} \parr !^l \lfloor B \rfloor^{\perp}, ?^l(!^l \lfloor A \rfloor^{\perp} \otimes !^r \lceil B \rceil^{\perp}) \\
(\wedge) & \vdash Cut_{G1m} \stackrel{\vdots}{\infty} \cdot \stackrel{\vdots}{l} \cdot \stackrel{\vdots}{r} \cdot \stackrel{\vdots}{f} \cdot \Uparrow !^l \lfloor A \rfloor^{\perp} \otimes !^l \lfloor B \rfloor^{\perp}, ?^l !^r \lceil A \rceil^{\perp} \parr ?^l !^r \lceil B \rceil^{\perp} \\
(\vee) & \vdash Cut_{G1m} \stackrel{\vdots}{\infty} \cdot \stackrel{\vdots}{l} \cdot \stackrel{\vdots}{r} \cdot \stackrel{\vdots}{f} \cdot \Uparrow !^l \lfloor A \rfloor^{\perp} \oplus !^l \lfloor B \rfloor^{\perp}, ?^l !^r \lceil A \rceil^{\perp} \,\&\, ?^l !^r \lceil B \rceil^{\perp} \\
(\forall) & \vdash Cut_{G1m} \stackrel{\vdots}{\infty} \cdot \stackrel{\vdots}{l} \cdot \stackrel{\vdots}{r} \cdot \stackrel{\vdots}{f} \cdot \Uparrow !^l \lfloor Bx \rfloor^{\perp}, ?^l \exists x. !^r \lceil Bx \rceil^{\perp} \\
(\exists) & \vdash Cut_{G1m} \stackrel{\vdots}{\infty} \cdot \stackrel{\vdots}{l} \cdot \stackrel{\vdots}{r} \cdot \stackrel{\vdots}{f} \cdot \Uparrow \exists x. !^l \lfloor Bx \rfloor^{\perp}, ?^l !^r \lceil Bx \rceil^{\perp}
\end{array}
$$

All these sequents have simple proofs. In general, deciding whether or not canonical systems are cut-coherent involves a simple algorithm (see Theorem 5.11).

Intuitively, the notion of cut-coherence on the meta-level corresponds to the property of reducing the complexity of a cut on the object-level. If a connective $\diamond$ is proven to have cut-coherent introduction rules, then a cut with formula $\diamond(\bar{x})$ can be replaced by simpler cuts using the operations of reductive cut-elimination, until atomic cuts are reached. This is proved by Theorem 5.8.

We need the following definition specifying cuts with atomic cut formulas only.

$$ACut = \exists A. Cut(A) \otimes atomic(A)).$$

Theorem 5.8
Let the disjoint union $\mathcal{X} \cup \{Cut\}$ be a principal, cut-coherent proof system. If $\vdash \mathcal{K} : \cdot \Uparrow \cdot$ is provable, then $\vdash A\mathcal{K} : \cdot \Uparrow \cdot$ is provable where $\mathcal{K}[\infty] = \mathcal{X} \cup \{Cut\}$ and $A\mathcal{K}[\infty] = \mathcal{X} \cup \{ACut\}$.

Proof. (Sketch – see [18] for the detailed proof.) The proof of this theorem follows the usual line of replacing cuts on general formulas for cuts on atomic formulas for first-order logic, being careful about the subexponentials. Let Ξ be a proof of the sequent $\vdash \mathcal{K} : \cdot \Uparrow \cdot$ ending with an object-level cut over a cut formula $\diamond(\bar{B})$ with left and right introduction rules

$$\exists \bar{x}(\lfloor \diamond(\bar{x}) \rfloor^{\perp} \otimes B_l) \quad \text{and} \quad \exists \bar{x}(\lceil \diamond(\bar{x}) \rceil^{\perp} \otimes B_r)$$

Since $\mathcal{X}$ is cut-principal, there exist proofs of $\vdash \mathcal{K}_1 : \cdot \Uparrow B_l$ and $\vdash \mathcal{K}_2 : \cdot \Uparrow B_r$, where $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$. Since $\mathcal{X}$ is a cut-coherent proof system theory the sequent $\vdash \mathcal{K}_\infty : \cdot \Uparrow \forall \bar{x}(B_l^{\perp} \parr B_r^{\perp})$ is provable. Thus, the following three sequents all have cut-free proofs in *SELL*[13]:

$$\vdash \mathcal{K}_1, B_l[\bar{B}/\bar{x}] \qquad \vdash \mathcal{K}_2, B_r[\bar{B}/\bar{x}] \qquad \vdash ?^{\infty} Cut, B_l[\bar{B}/\bar{x}]^{\perp}, B_r[\bar{B}/\bar{x}]^{\perp}$$

By using two instances of *SELL* cut, we can conclude that[14]

$$\vdash \mathcal{K}_1, \mathcal{K}_2$$

has a proof with cut. Applying the cut-elimination process for *SELL* will yield a cut-free *SELL* proof of the same sequent. Observe that the elimination process can only instantiate eigenvariables of the proof with "simpler" formulas, hence the sizes of object-level cut formulas in the resulting cut-free meta-level proof does not increase. Using the completeness of *SELL* in *SELLF* we know that

$$\vdash \mathcal{K} : \cdot \Uparrow \cdot$$

has a proof of smaller object-level cuts and the result follows by induction. ∎

The last step in Gentzen's cut-elimination strategy is to eliminate atomic cuts by permuting them upwards. However, as in the transformation of proofs with cuts into proofs with principal cuts only, the subexponential bangs may disallow that atomic cuts can be eliminated. A further restriction on cut clauses is needed.

Definition 5.9
Let $\mathcal{X}$ be a principal, cut-coherent proof system theory. We say that a cut clause $Cut = \exists A.!^a ?^b \lfloor A \rfloor \otimes !^c ?^d \lceil A \rceil$ is *weak* if for all $s, t \in I$ such that $?^s \lfloor \cdot \rfloor, ?^t \lceil \cdot \rceil$ appears in $\mathcal{X}$, $b \preceq s$ and $d \preceq t$.

$\mathcal{X}$ is called *weak cut-coherent* if, for all $Cut \in \mathcal{X}$, Cut is weak.

Theorem 5.10
Let the disjoint union $\mathcal{X} \cup \{ACut\}$ be a weak cut-coherent proof system. Let $\Gamma_o \longrightarrow \Delta_o$ be an object-level sequent and $\vdash \mathcal{K} : \cdot \Uparrow \cdot$ be its *SELLF* encoding, where $\mathcal{K}[\infty] = \mathcal{X} \cup \{ACut\}$. If $\vdash \mathcal{K} : \cdot \Uparrow \cdot$ is provable, then $\vdash \mathcal{K}' : \cdot \Uparrow \cdot$ is provable where $\mathcal{K}'[\infty] = \mathcal{X}$ and $\mathcal{K}[i] = \mathcal{K}'[i]$ for any other $i \in I$.

Proof. The usual proof that permutes an atomic cut up in a proof can be applied here (since the system is principal). Any occurrence of an instance of $[D_\infty]$ on the *ACut* formula can be moved up in a proof until it can either be dropped entirely or until one of the premises is

[13] By abuse of notation, we will represent the contexts in *SELLF* and its translation in *SELL* using the same symbol.

[14] Reminding that $Cut \in \mathcal{K}[\infty]$.

proved by an instance of $[D_\infty]$ on the *Init*:[15]

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Xi\\ \vdash \mathcal{K}_1 \leq_a +_b \lfloor A\rfloor : \cdot \Uparrow \cdot\end{array}}{\vdash \mathcal{K}_1 : \cdot \Downarrow\, !^a ?^b \lfloor A\rfloor}\,[!^a, ?^b] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K}_2^1 : \cdot \Downarrow \lceil A\rceil^\perp} \quad \dfrac{}{\vdash \mathcal{K}_2^2 : \cdot \Downarrow \lfloor A\rfloor^\perp}}{\vdash \mathcal{K}_2 \leq_c +_d : \cdot \Downarrow \lceil A\rceil^\perp \otimes \lfloor A\rfloor^\perp}\,[\otimes]}{\vdash \mathcal{K}_2 \leq_c +_d \lceil A\rceil : \cdot \Uparrow \cdot}\,[D_\infty, \exists]}{\vdash \mathcal{K}_2 : \cdot \Downarrow\, !^c ?^d \lceil A\rceil}\,[!^c, ?^d]}{\vdash \mathcal{K}_1 \otimes \mathcal{K}_2 : \cdot \Downarrow\, !^a ?^b \lfloor A\rfloor \otimes\, !^c ?^d \lceil A\rceil}\,[\otimes]}{\vdash \mathcal{K} : \cdot \Uparrow \cdot}\,[D_\infty, \exists]$$

In that case, there must exist an index s such that $\lfloor A\rfloor \in \mathcal{K}_2^2[s]$. If $b \leq s$, then we can substitute the proof of the conclusion of the cut inference above by the proof Ξ (similar to the right case). Hence the result holds for weak cut-coherent systems. ∎

The next result states that to check whether or not a proof system encoding is weak cut-coherent is decidable. See [18] for a similar proof.

Theorem 5.11
Determining whether or not a canonical proof system is weak cut-coherent is decidable. In particular, determining if the cut clause proves the duality of the introduction rules for a given connective can be achieved by proof search in *SELLF* bounded by the depth $v + 2$ where v is the maximum number of premise atoms in the bodies of the introduction clauses.

We can develop a general method for checking whether a proof system encoded in *SELLF* admits cut-elimination by putting all these results together. The first step is to use Lemma 5.2 to check for which clauses the cut permutes over. Then for each remaining clause, C, check using Lemma 5.4, the introduction/structural clauses of the system permutes over C. After this step one is reduced with the non-trivial cases for when the transformation of a proof with cuts into a proof with atomic cuts only is not straightforward and must be proved individually. We then check whether the theory is cut-coherent, which from Theorem 5.8, implies that principal cuts can be reduced to atomic cuts. This check requires bounded proof search as described in Theorem 5.11. Finally, we check whether atomic cuts can be eliminated by checking whether the theory is weak cut-coherent. We have implemented this method, as well as the checking for atomic identities, as detailed in Section 6.

5.2 Atomic Identities

The notion of cut-coherence implies that non-atomic principle cuts can be replaced by simpler ones. We now consider the dual problem of replacing initial axioms with its atomic version. The discussion bellow is pretty much similar to the ideas presented in [18].

Definition 5.12
Let $\mathcal{X}$ be a canonical proof system theory and $\diamond$ an object-level connective of arity $n \geq 0$. Furthermore, let the formulas

$$\exists \bar{x}(\lfloor \diamond(\bar{x})\rfloor^\perp \otimes B_l) \quad \text{and} \quad \exists \bar{x}(\lceil \diamond(\bar{x})\rceil^\perp \otimes B_r)$$

be the left and right introduction rules for $\diamond$, where the free variables of B_l and B_r are in the list of variables $\bar{x}$. The object-level connective $\diamond$ has *initial-coherent introduction rules* if the

[15] Here A is an atomic object level formula.

sequent $\vdash \mathcal{K}_\infty : \cdot \Uparrow \forall\bar{x}(?^\infty B_l \parr ?^\infty B_r)$ is provable in $SELLF$, where $\mathcal{K}_\infty[\infty] = \{Init\}$ and $\mathcal{K}_\infty[i] = \emptyset$ for any other $i \in I$. A canonical proof system theory is called *initial-coherent* if all object-level connectives have initial-coherent introduction rules.

It is easy to see that determining initial-coherency is simple and that initial coherency does not imply cut-coherency (and vice-versa). In general, we take both of these coherence properties together.

Definition 5.13
A cut-coherent theory that is also initial-coherent is called a *coherent theory*.

Proposition 5.14
Let $\mathcal{X}$ be a coherent theory and $\diamond$ an object-level connective of arity $n \geq 0$. Furthermore, let the formulas

$$\exists\bar{x}(\lfloor\diamond(\bar{x})\rfloor^\perp \otimes B_l) \quad \text{and} \quad \exists\bar{x}(\lceil\diamond(\bar{x})\rceil^\perp \otimes B_r)$$

be the left and right introduction rules for $\diamond$. Then B_r and B_l are dual formulas in $SELLF$.

Proof. From the definition of cut-coherent, B_l entails B_r in a theory containing $\{Cut\}$. Similarly, from the definition of initial-coherence, B_r entails B_l in a theory containing $Init$. Thus, the equivalence $B_r \equiv B_l$ is provable in a theory containing $\{Cut\}$ and $Init$. Hence B_r and B_l are duals. ∎

Finally, the next theorem states that, in coherent systems, the initial rule can be restricted to its atomic version. For this theorem, we need to axiomatize the meta-level predicate $atomic(\cdot)$. This axiomatization can be achieved by collecting into the theory Δ all formulas of the form $\exists\bar{x} : (atomic(p(x1; :::; xn)))^\perp$ for every predicate of the object logic.

For the next theorem, we also need the following definition

$$AInit = \exists A.Init(A) \otimes atomic(A)).$$

Theorem 5.15
Given an object level formula B, let $Init(B)$ denote the formula $\lfloor B\rfloor^\perp \otimes \lceil B\rceil^\perp$, let Δ be the theory that axiomatizes the meta-level predicate $atomic(\cdot)$, $\mathcal{X} \cup Init$ be a coherent proof theory and $\mathcal{K}_\infty = \{\mathcal{X}, AInit, \Delta\}$. Then the sequent $\vdash \mathcal{K}_\infty : \cdot \Uparrow Init(B)$ is provable.

5.3 Invertibility of rules

Another property that has been studied in the sequent calculus setting is the invertibility of rules. We say that a rule is invertible if the provability of the conclusion sequent implies the provability of all the premises.

This property is of interest to proof search since invertible rules permute down with the other rules of a proof, reducing hence proof-search non-determinism. In particular, in systems with only invertible rules, the bottom-up search for a proof can stop as soon as a non provable sequent is reached.

For example, it is well known that all rules in G3c (see [29]) are invertible. This system is specified in Figure 13. Observe that the meta level connectives in the bodies are negative. Therefore, its introduction rule is specified using only invertible focused rules. The following is a straightforward result, as all the connectives appearing in a monopole are negative.

Theorem 5.16
A monopole introduction clause corresponds to an invertible object level rule.

$(\Rightarrow L)$	$\lfloor A \Rightarrow B \rfloor^{\perp} \otimes ?^{r}\lceil A \rceil \mathbin{\&} ?^{l}\lfloor B \rfloor$	$(\Rightarrow R)$	$\lceil A \Rightarrow B \rceil^{\perp} \otimes ?^{l}\lfloor A \rfloor \parr ?^{r}\lceil B \rceil$
$(\wedge L)$	$\lfloor A \wedge B \rfloor^{\perp} \otimes ?^{l}\lfloor A \rfloor \parr ?^{l}\lfloor B \rfloor$	$(\wedge R)$	$\lceil A \wedge B \rceil^{\perp} \otimes ?^{r}\lceil A \rceil \mathbin{\&} ?^{r}\lceil B \rceil$
$(\vee R)$	$\lceil A \vee B \rceil^{\perp} \otimes ?^{r}\lceil A \rceil \parr ?^{r}\lceil B \rceil$	$(\vee L)$	$\lfloor A \vee B \rfloor^{\perp} \otimes ?^{l}\lfloor A \rfloor \mathbin{\&} ?^{l}\lfloor B \rfloor$

FIG. 13. Specification of G3c.

6 Implementation

We have implemented a tool that takes a *SELLF* specification of a proof system and checks automatically whether the proof system admits cut-elimination and whether the system with atomic initials is complete. Our tool is implemented in OCaml and there is an online version with some examples at `http://www.logic.at/people/giselle/tatu`. The specification of proof systems is done as described in Section 3. In particular, the clauses specifying a proof system are separated into four parts: introduction clauses, structural clauses, cut clauses, and the identity clauses. We have written the specification of all the systems described in Section 4.

The tool also contains the machinery necessary for checking the conditions described in Section 5. It implements the static analysis described in Lemmas 5.2 and 5.4. As detailed at the end of Section 5.1, the tool determines cases for when the cut rule can permute over other introduction rules and for when an introduction rule permutes over another introduction rule. Whenever some clauses of the encoding does not satisfy such criteria, then it outputs an error message. Detecting corner cases can be useful for detecting design flaws in the specification of a proof system. For the systems $G1m$ and Lax, our tool was able to check that indeed a proof with cuts can be transformed into a proof with principal cuts only. For the other systems, it identified some permutations by vacuously that it could not prove automatically. However, these can be easily checked manually.

For checking whether an encoding is cut-coherent, our tool performs bounded proof search, where the bound is detemined as described in Theorem 5.11. In order to handle the problem of context splitting during proof search, our tool implements the lazy splitting detailed in [4] for linear logic. The method easily extends to *SELLF*. Another difference, however, is that our system is one-sided classical logic. Therefore, we do not implement the back-chaining style proof search used in [4], but rather proof search based on the focused discipline described in Section 2. Furthermore, as previously mentioned, proof search is bounded by the height of derivations, measured by the number of decide rules. This is enough for checking whether an encoding is cut-coherent. In a similar fashion, the tool also checks by using bounded proof search whether the encoded proof system is complete when using atomic initial rules by checking whether the system is initial coherent (see Definition 5.12). For all the examples that we have implemented, our tool checks all the conditions described above in less than a second.

7 Related Work

The present work has its foundations on the works [22, 18] by Miller, Nigam, and Pimentel, where plain linear logic was used as the framework for specifying sequent systems, and reasoning about them. The motivation for the generalization proposed here was based initially on the fact that there are a number of proof systems that can be encoded *SELLF* but cannot be encoded in the same declarative fashion (such as without mentioning side-formulas) in linear logic without subexponentials. Moreover, the encodings in [18] are only on the level of proofs and not on the level of derivations [22]. Therefore, proving adequacy in [18] involves more complicated techniques than the simple proofs by induction on the height of focused derivations used here. Finally, when trying to deal with the verification using *SELLF*, we ended up being able to propose more general conditions for permutation of clauses, which enabled more general criteria for proving cut-elimination of systems.

It turns out that specification and verification of proof systems is a very important branch of the proof theory field. In fact, there exists a number of works willing to provide adequate tools for dealing with systems in a general and yet natural way, making it possible then to use the rich meta-theory proposed in order to reason about the specifications. For example, Pfenning proposed a method of proving cut-elimination [24] from specifications in intuitionistic linear logic. This method has been applied to a number of proof systems and implemented by using the theorem prover Twelf [25]. For instance, the encoding of Lax logic and its cut-elimination proof can be found at `http://twelf.org/wiki/Lax_logic`. It happens that this procedure is only semi-automated, in the sense that, for any given proof system, one has to prove all the permutation lemmas and reductions needed in the cut-elimination from scratch.

In the present paper, we adopted a more uniform approach, establishing general criteria to the specification for proving properties of the specified systems. Since we are dealing with classical linear logic (where negation is involutive), our encodings never mention side-formulas, only the principal formulas of the rules. Such declarative specifications produce not only clean and natural encodings, but it also allows for easy meta-level reasoning.

Ciabattoni and Terui in [6] have proposed a general method for extracting cut-free sequent calculus proof systems from Hilbert style proof systems. Their method can be used for a number of non-trivial logics, including intuitionistic linear logic extended with knotted structural rules. However, a main difference to our work is that they do not provide a decision criteria for when a system falls into their framework. On the other hand, we do not provide means to encode Hilbert style proof systems. It seems that our methods are complementary and can be combined, so to enable the specification of Hilbert style proof systems as well as reason over them. However, the challenges of integrating these methods have still to be investigated.

Checking whether a rule permutes over another was also topic of the recent work [14]. As in our approach, Lutovac and Harland investigate syntactic conditions which allow to check the validity of such permutations. A number of cases of permutations and examples are provided. A main difference to our approach is that we fixed the specification language, namely *SELLF*, to specify inference rules and proof systems, whereas [14] does not make such commitment. On one hand, we can only reason about systems "specifiable" in *SELLF*, but on the other hand, the use a logical framework allows for the construction of a general tool that can check for permutations automatically. It is not yet clear how one could construct a similar tool using the approach in [14].

8 Conclusions and Future Work

In this paper, we showed that it is possible to specify a number of non-trivial structural properties by using subexponential connectives. In particular, we demonstrated that it is possible to specify proof systems whose sequents have multiple contexts that are treated as multisets or sets. Moreover, it is possible to specify inference rules that require some formulas to be weakened and inference rules that require some side-formula to be present in its conclusion. We have also introduced the machinery for checking whether encoded proof systems have three important properties, namely the admissibility of the cut rule, the completeness of atomic identity rules, and the invertibility of rules. Finally, we have also build an implementation that automatically checks some of these criteria.

There are a number of directions to follow from this work. As argued in the paper, a main challenge for determining whether a proof system admits cut-elimination by just checking its specification is checking whether a rule permutes over another one. Although we found general conditions that apply to many systems, these criteria are static, that is, it is enough to just inspect the specification without executing it. It seems possible to check for more permutations by performing bounded proof search, similar to what was done for checking the cut-coherence property. In particular, we are investigating how to use existing propositional solvers together with bounded proof search to perform this check automatically.

Another future direction is of investigating the role of the polarity of atomic meta-level formulas in the specification of proof systems using *SELLF*. [22] showed in a linear logic setting that a number of proof systems can be faithfully encoded by playing with the polarity of atomic formulas. Here, we assigned to all atomic formulas a negative polarity, but this choice is not enforced by the completeness of the focusing strategy (see [19]). In fact, a different (global) assignment for atoms could be chosen. However, to use such a technique here would imply a change on the definition of bipoles, as with the current definition polarities would play a very limited role because all atomic formulas are in the scope of a subexponential question-mark. We are investigating alternative definitions, so that we can still use subexponentials in a sensible way and at the same time play with the polarity of atomic formulas.

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